On the Perimeter of an Ellipse

Paul Abbott

Computing accurate approximations to the perimeter of an ellipse is a favorite problem of mathematicians, attracting luminaries such as Ramanujan [1, 2, 3]. As is well known, the perimeter $P$ of an ellipse with semimajor axis $a$ and semiminor axis $b$ can be expressed exactly as a complete elliptic integral of the second kind.

What is less well known is that the various exact forms attributed to Maclaurin, Gauss-Kummer, and Euler are related via quadratic hypergeometric transformations. These transformations lead to additional identities, including a particularly elegant formula symmetric in $a$ and $b$.

Approximate formulas can, of course, be obtained by truncating the series representations of exact formulas. For example, Kepler used the geometric mean, $P \approx 2\pi \sqrt{ab}$, as a lower bound for the perimeter. In this article, we examine the properties of a number of approximate formulas, using series methods, polynomial interpolation, rational polynomial approximants, and minimax methods.

Introduction

The well-known formula for the perimeter $P$ of an ellipse with semimajor axis $a$ and semiminor axis $b$ can be expressed exactly as a complete elliptic integral of the second kind, which can also be written as a Gaussian hypergeometric function,

$$ P = 4a \, E \left( 1 - \frac{b^2}{a^2} \right) = 2\pi \, a \, _2F_1 \left( \frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{b^2}{a^2} \right). $$

The quadratic hypergeometric transformations [4, 5] lead to additional identities, including a particularly elegant formula, symmetric in $a$ and $b$,

$$ P = 2\pi \sqrt{ab} \, P_\frac{1}{2} \left( \frac{a^2 + b^2}{2ab} \right), $$

where $P_\nu(z)$ is a Legendre function.
Cartesian Equation

The Cartesian equation for an ellipse with center at \((0, 0)\), semimajor axis \(a\), and semiminor axis \(b\) reads

\[
\text{In[1]:=} \quad E(x_, y_) = \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1;
\]

Introducing the parameter \(\varphi\) into the Cartesian coordinates, as \((x = a \sin(\varphi), \quad y = b \cos(\varphi))\), we verify that the ellipse equation is satisfied.

\[
\text{In[2]:=} \quad \text{Simplify}[E(a \sin(\varphi), \ b \cos(\varphi))]
\]

\[
\text{Out[2]} = \text{True}
\]

Arclength

In general, the parametric arclength is defined by

\[
\mathcal{L} = \int_{\varphi_1}^{\varphi_2} \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} \ d\varphi. \tag{1}
\]

The arclength of an ellipse as a function of the parameter \(\varphi\) is an (incomplete) elliptic integral of the second kind.

\[
\text{In[3]:=} \quad \mathcal{L}(\varphi) = \text{With}[\{x = a \sin(\varphi), \ y = b \cos(\varphi)\}, \quad \text{Simplify}\left[\int_{\varphi_1}^{\varphi_2} \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2} \ d\varphi, \ a > b > 0 \land 0 < \varphi < \frac{\pi}{2}\}\right]
\]

\[
\text{Out[3]} = \quad a \ E\left(\varphi \mid 1 - \frac{b^2}{a^2}\right)
\]

Since

\[
\text{In[4]:=} \quad \mathcal{L}(0) = 0
\]

\[
\text{Out[4]} = \text{True}
\]

the arclength of the ellipse is

\[
\mathcal{L}(\varphi) = a \ E(\varphi \mid e^2), \tag{2}
\]

where the eccentricity \(e\) is defined by

\[
\text{In[5]:=} \quad e(a_, b_) = \sqrt{1 - \frac{b^2}{a^2}}.
\]
Perimeter

Since the parameter ranges over $0 \leq \varphi \leq \frac{\pi}{2}$ for one quarter of the ellipse, the perimeter of the ellipse is

$$\text{In}[6]:= \mathcal{P}_1(a_\_, b_\_) = 4 \mathcal{L}\left(\frac{\pi}{2}\right)$$

$$\text{Out}[6]= 4 a E\left(1 - \frac{b^2}{a^2}\right)$$

That is, $\mathcal{P} = 4 a E(e^2)$, where $E(m)$ is the complete elliptic integral of the second kind.

Alternative Expressions for the Perimeter

The given expression for the perimeter of the ellipse is *unsymmetrical* with respect to the parameters $a$ and $b$. This is “unphysical” in that both parameters, being lengths of the (major and minor) axes, should be on the same footing. We can expect that a *symmetric* formula, when truncated, will more accurately approximate the perimeter for both $a \geq b$ and $a \leq b$.

Noting that the complete elliptic integral is a Gaussian hypergeometric function,

$$\text{In}[7]:= \text{Hypergeometric2F1}\left(\frac{1}{2}, -\frac{1}{2}; 1; z\right)$$

$$\text{Out}[7]= \frac{2 E(z)}{\pi}$$

we obtain Maclaurin’s 1742 formula [2]

$$\text{In}[8]:= \mathcal{P}_1(a, b) = 2 \pi a \text{Hypergeometric2F1}\left(\frac{1}{2}, -\frac{1}{2}; 1; e(a, b)^2\right)$$

$$\text{Out}[8]= \text{True}$$

Equivalent alternative expressions for the perimeter of the ellipse can be obtained from quadratic transformation formulas for Gaussian hypergeometric functions. For example, using functions.wolfram.com/07.23.17.0106.01,
\begin{align*}
\text{In[9]} &= \text{Simplify}\left[_{2}F_{1}(\alpha, \beta; 2 \beta; z) = \frac{2F_{1}\left(\alpha - \beta + \frac{1}{2}; \beta + \frac{1}{2}; \left(\frac{1 - \sqrt{1 - z}}{\sqrt{1 - z} + 1}\right)^{2}\right)}{\left(\frac{1}{2}\left(\sqrt{1 - z} + 1\right)\right)^{2\alpha}}\right].
\end{align*}

\begin{align*}
\{\beta \to \frac{1}{2}, \alpha \to -\frac{1}{2}, z \to e(a, b)^{2}\}, & a > b > 0 \\
\text{Out[9]} &= 4a E\left(1 - \frac{b^{2}}{a^{2}}\right) = (a + b) \pi \frac{\Gamma}{2}\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(a - b)^{2}}{(a + b)^{2}}\right)
\end{align*}

and noting that
\begin{align*}
\text{In[10]} &= \text{Simplify}\left[\frac{(a - b)^{2}}{(a + b)^{2}} = 1 - \frac{4a b}{(a + b)^{2}}\right]
\end{align*}

\text{Out[10]} = \text{True}

we obtain the following symmetric formula
\begin{align*}
\text{In[11]} &= \mathcal{P}_{2}(a, b) = \pi (a + b) \frac{\Gamma}{2}\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{4a b}{(a + b)^{2}}\right);
\end{align*}

first obtained by Ivory in 1796, but known as the Gauss-Kummer series [2].

Introducing the homogeneous symmetric parameter \( b = \frac{(a-b)^{2}}{(a+b)^{2}} = 1 - \frac{4a b}{(a+b)^{2}} \), we have (cf. mathworld.wolfram.com/Ellipse.html)
\begin{align*}
\text{In[12]} &= \text{Simplify}\left[\text{FunctionExpand}\left[\pi (a + b) \frac{\Gamma}{2}\left(-\frac{1}{2}, -\frac{1}{2}; 1; b\right)\right]\right]
\end{align*}

\text{Out[12]} = 2(a + b) (2E(b) + (b - 1) K(b))

Explicitly, the Gauss-Kummer series reads
\begin{align*}
\text{In[13]} &= \mathcal{P}_{3}(a, b) = \text{FullSimplify}\left[\text{FunctionExpand}\left[\mathcal{P}_{2}(a, b), a > b > 0\right]\right]
\end{align*}

\text{Out[13]} = 4(a + b) E\left(1 - \frac{4a b}{(a + b)^{2}}\right) - \frac{8a b K\left(1 - \frac{4a b}{(a+b)^{2}}\right)}{a + b}

Instead, using functions.wolfram.com/07.23.17.0103.01, we obtain Euler’s 1773 formula (see also [2])
\begin{align*}
\text{In[14]:=} & \quad \text{Simplify}\left[ \binom{1}{2} F_1(\alpha, \beta; 2 \beta; z) = \frac{2 F_1\left(\alpha, \frac{\alpha+1}{2}; \beta + \frac{1}{2}, \frac{z^2}{(2-z)^2}\right)}{(1 - \frac{z}{2})^\alpha} \right] /.
\end{align*}

\begin{align*}
\text{Out[14]=} & \quad \left\{ \beta \rightarrow \frac{1}{2}, \alpha \rightarrow -\frac{1}{2}, z \rightarrow -\frac{z}{\beta} \right\}
\end{align*}

The hidden symmetry with respect to the interchange \( a \leftrightarrow b \) is revealed.

\text{In[15]:=} \quad \text{FullSimplify}[\% , b > a > 0]

\text{Out[15]=} \quad b \left( 1 - \frac{a^2}{b^2} \right) = a \left( 1 - \frac{b^2}{a^2} \right)

Defining

\text{In[16]:=} \quad P_4(a_-, b_-) = \pi \sqrt{2 (a^2 + b^2)} \binom{1}{2} F_1\left(\frac{1}{4}, -\frac{1}{4}; 1; \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2\right);

we can directly check the formula.

\text{In[17]:=} \quad \text{Simplify}[\text{FunctionExpand}[P_4(a, b) = P_1(a, b)], a > b > 0]

\text{Out[17]=} \quad \text{True}

\section*{Other Identities}

There are many other possible transformation formulas that can be applied to obtain alternative expressions for the perimeter. For example, using functions.wolfram.com/07.23.17.0054.01 we obtain the following formula

\text{In[18]:=} \quad P_5(a_-, b_-) = P_2(a, b) /.
2 F_1(a_-, b_-, c_-, z_-) \rightarrow (1 - z)^{-a - b - c} \binom{3}{2} F_1(c - a, c - b; c; z)

\text{Out[18]=} \quad 16 a^3 b^3 \frac{\pi}{(a+b)^3} F_1\left(\frac{3}{2}, \frac{3}{2}; 1; 1 - \frac{4 a b}{(a+b)^2}\right)

The perimeter can also be expressed in terms of Legendre functions (see Sections 8.13 and 15.4 of [6]). For example, using 15.4.15 of [6] we obtain the elegant and simple symmetric formula
\[ P_6(a, b) = \]
\[ \text{Simplify}\left[ P_2(a, b) \right. \cdot 2 F_1(a, b; c; x) : \Gamma(a - b + 1)(1 - x)^{-b}(x)^{b-a} \]
\[ p_{b-a} \left( \frac{x + 1}{1 - x} \right) \right/ c = a - b + 1, a > 0 \land b > 0 \]

\[ \text{Out[19]} = 2 \sqrt{a b} \pi P_1 \left( \frac{a^2 + b^2}{2 a b} \right) \]

Alternatively, this result follows directly from 8.13.6 of [6] with \( e^\eta = \frac{a}{b} \Rightarrow \cosh(\eta) = \frac{a^2 + b^2}{2 a b} \). This form can be used to prove that the perimeter of an ellipse is a homogenous mean (cf. [7]), extending the arithmetic-geometric mean (AGM) already used as a tool for computing elliptic integrals [8].

Using functions.wolfram.com/07.07.26.0001.01 gives yet another formula involving complete elliptic integrals.

\[ P_7(a, b) = \]
\[ \text{Simplify}\left[ \text{FunctionExpand}\left[ P_6(a, b) \right. \cdot P_\infty(z) \right. \rightarrow 2 F_1 \left( -v, v + 1; 1; \frac{1 - z}{2} \right) \right. \]
\[ \text{Out[20]} = 4 \sqrt{a b} \left( 2 E \left( -\frac{(a - b)^2}{4 a b} \right) - K \left( -\frac{(a - b)^2}{4 a b} \right) \right) \]

\[ \square \text{ Comparisons} \]

Here we compare the seven formulas for \( b = 2 a \),

\[ \text{Out[21]} = \left\{ 4 a E(-3), 3 a \pi 2 F_1 \left( \frac{1}{2}, -\frac{1}{9}; 1; \frac{1}{9} \right), \frac{4}{3} a \left( 9 E \left( \frac{1}{9} \right) - 4 K \left( \frac{1}{9} \right) \right), \right. \]
\[ \sqrt{10} a \pi 2 F_1 \left( \frac{1}{4}, -\frac{1}{25}; \frac{3}{2}, 1; \frac{3}{2} \right), \frac{64}{27} a \pi 2 F_1 \left( \frac{3}{2}, -\frac{1}{2}; 1; \frac{1}{9} \right), \]
\[ 2 \sqrt{2} a \pi P_1 \left( \frac{5}{4} \right), 4 \sqrt{2} a \left( 2 E \left( -\frac{1}{8} \right) - K \left( -\frac{1}{8} \right) \right) \right. \]
\[ \text{Out[22]} = \{ 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a \} \]

\[ \text{Out[23]} = \text{True} \]

and for \( b = \frac{a}{3} \).
\[\text{In[24]}:= \text{Simplify}\left\{P_1\left(a, \frac{a}{3}\right), P_2\left(a, \frac{a}{3}\right), P_3\left(a, \frac{a}{3}\right), P_4\left(a, \frac{a}{3}\right), P_5\left(a, \frac{a}{3}\right), P_6\left(a, \frac{a}{3}\right), P_7\left(a, \frac{a}{3}\right)\right\}, a > 0\]

\[\text{Out[24]}= \begin{array}{l}
4 a E_1\left(\frac{8}{9}, \frac{4}{3} a \pi F_1\left(\frac{-1}{4}, -\frac{1}{4}; 1; \frac{1}{4}\right)\right), \\
\frac{2}{3} a \left(8 E\left(\frac{1}{4}\right) - 3 K\left(\frac{1}{4}\right)\right), \\
\frac{2}{3} \sqrt{5} a \pi F_1\left(\frac{1}{4}, -\frac{1}{4}; 1; \frac{16}{25}\right), \\
\frac{3}{4} a \pi F_1\left(\frac{3}{2}, -\frac{3}{2}; 1; \frac{1}{4}\right), \\
2 a \pi F_1\left(\frac{3}{4}, -\frac{3}{4}; 1; \frac{1}{4}\right) - 4 K\left(\frac{3}{4}\right)\end{array}\]

\[\text{In[25]}:= \text{N}\left[\%ight]\]

\[\text{Out[25]}= \{4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a, 4.45496 a\}\]

\[\text{In[26]}:= \text{Equal @@ \%}\]

\[\text{Out[26]}= \text{True}\]

\section*{Numerical Approximation}

At [1] we are encouraged to search for “…an efficient formula using only the four algebraic operations (if possible, avoiding even square-root) with a maximum error below 10 ppm. It would also be nice if such a formula were exact for both the circle and the degenerate flat ellipse”.

The Gauss-Kummer series expressed as a function of the homogeneous variable

\[b = 1 - \frac{4 a b}{(a+b)^2}\]

reads

\[\text{In[27]}:= \text{GaussKummer}[h_] = \frac{P_2(a, b)}{a + b} / a + b \rightarrow \frac{2 \sqrt{a b}}{\sqrt{1 - b}}\]

\[\text{Out[27]}= \pi F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; b\right)\]

\section*{Series Expansions}

The series expansion about \(b = 0\) is useful for small \(b\).
GaussKummer[b] + O(b)^9

\[\begin{align*}
\text{Out[28]} &= \pi \left( 1 + \frac{\pi b}{4} + \frac{\pi b^2}{64} + \frac{\pi b^3}{256} + \frac{25 \pi b^4}{16384} + \frac{49 \pi b^5}{65536} + \frac{441 \pi b^6}{1048576} + \frac{1089 \pi b^7}{4194304} + \frac{184041 \pi b^8}{1073741824} + O(b^9) \right) \\
\text{Around } b = 1, \text{ terms in } \log(1 - b) \text{ arise.}
\end{align*}\]

Simplify[Series[GaussKummer[b], {b, 1, 2}], 0 < b < 1]

\[\begin{align*}
\text{Out[29]} &= 4 + (b - 1) + \frac{1}{16} \left( -2 \log(1 - b) - 4 \psi(0)\left(\frac{3}{2}\right) - 4 \gamma + 3 \right) (b - 1)^2 + O((b - 1)^3)
\end{align*}\]

Using functions.wolfram.com/07.23.06.0015.0, we obtain the general term of this series (c.f. 17.3.33 through 17.3.36 of [6]),

Simplify[GaussKummer[b] /. \[2F1(a_, b_; c_; z_) \to With[n = c - a - b], \frac{(n - 1)! \Gamma(a + b + n)}{\Gamma(a + n) \Gamma(b + n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k (1 - z)^k}{k! (1 - n)_k} + \frac{\Gamma(a + b + n)}{\Gamma(a) \Gamma(b)} \frac{1}{\sum_{k=0}^{\infty} \frac{1}{k! (k + n)!}} ((a + n)_k (b + n)_k) \left( -\log(1 - z) + \psi(k + 1) + \psi(k + n + 1) - \psi(a + k + n) - \psi(b + k + n) \right) (1 - z)^k (z - 1)^n \right]]

\[\begin{align*}
\text{Out[30]} &= \frac{1}{4} \left( \sum_{k=0}^{\infty} \frac{(1 - b)^k \left(\frac{1}{2}\right)_k^2 \left( -\log(1 - b) + \psi(0)(k + 1) + \psi(0)(k + 3) - 2 \psi(0)\left(k + \frac{3}{2}\right) \right)}{k! (k + 2)!} \right) \\
&= (b - 1)^2 + 4 (b + 3)
\end{align*}\]

\[\square\text{ Polynomial Approximants}\]

\[\text{Linear Approximant}\]

From the exact values at b = 0,

GaussKummer[0]

\[\text{Out[31]} = \pi\]

and at b = 1,

GaussKummer[1]

\[\text{Out[32]} = 4\]
we construct the linear *extreme perfect* approximant.

\[ \text{In[33]} = \text{Linear}[h_] = \text{Simplify}[(1 - b) \text{GaussKummer}[0] + b \text{GaussKummer}[1]] \]

\[ \text{Out[33]} = \pi - b (\pi - 4) \]

\[ \text{In[34]} = \text{Plot}[[\text{GaussKummer}[b], \text{Linear}[b]], \{b, 0, 1\}] \]

**Quadratic Approximant**

The quadratic approximant, exact at \( b = 0, \frac{1}{2}, 1, \)

\[ \text{In[35]} = \text{FullSimplify}[[\text{Table}[[b, \text{GaussKummer}[b]], \{b, 0, 1, \frac{1}{2}\}]]] \]

\[ \text{Out[35]} = \left(\begin{array}{c} 0 \\ \frac{\pi}{2} \\ 1 \\ 4 \end{array}\right) \]

\[ \text{In[36]} = \text{Quadratic}[h_] = N[\text{InterpolatingPolynomial}[%, b]] \]

\[ \text{Out[36]} = (0.0891819 (b - 0.5) + 0.813816) b + 3.14159 \]

has a maximum absolute relative error less than \( 8 \times 10^{-4} \).

\[ \text{In[37]} = \text{Plot}[[10^4 (1 - \frac{\text{Quadratic}[b]}{\text{GaussKummer}[b]}), \{b, 0, 1\}] \]

\[ \text{Out[37]} = -8 \sim -2 \]

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\textbf{$n^{\text{th}}$-order Polynomial Approximant}

Here is the $n^{\text{th}}$-order “even-tempered” polynomial approximant, exact at $b = \frac{m}{n}$ for $m = 0, 1, \ldots, n$.

\begin{verbatim}
In[38]:= poly[n_] := poly[n] = Function[b, Evaluate[InterpolatingPolynomial[N[Table[{b, GaussKummer[b]}, {b, 0, 1, 1/n}], b]]]]
\end{verbatim}

The $9^{\text{th}}$-order approximant has a maximum absolute relative error less than $10^{-5}$.

\begin{verbatim}
In[39]:= Plot[10^6 \left(1 - \frac{poly[9][b]}{GaussKummer[b]}\right),
\{b, 0, 1\}, PlotRange -> All, PlotPoints -> 30]
\end{verbatim}

\textbf{Chebyshev Polynomial Approximant}

Sampling the Gauss-Kummer function at the zeros of $T_n(2x - 1)$, which are at $x_m = \cos\left(\left(m + \frac{1}{4}\right)\frac{\pi}{n}\right)$, yields a Chebyshev polynomial approximant.

\begin{verbatim}
In[40]:= Chebyshevpoly[n_] :=
Chebyshevpoly[n] = Function[b, Evaluate[InterpolatingPolynomial[N[Join[{0, GaussKummer[0]}, Table[{\cos^2\left(\left(m + \frac{1}{4}\right)\frac{\pi}{n}\right)}, {m, n}]], b]]]]
\end{verbatim}

The $8^{\text{th}}$-order approximant has a maximum absolute relative error less than $7 \times 10^{-6}$.
\[\text{In[41]}:= \text{Plot}\left[10^6 \left(1 - \frac{\text{ChebyshevPoly}[8][b]}{\text{GaussKummer}[b]}\right), \{b, 0, 1\}, \text{PlotRange} \rightarrow \text{All}\right]\]

\[\text{Out[41]}= \text{Plot} \text{Graph}\]

\[\text{In[42]}:= \text{<< FunctionApproximations`}\]

we obtain a family of \([N, M]\) rational polynomial minimax approximations.

\[\text{In[43]}:= \text{GKapprox}[n_\_ , m_\_] := \text{GKapprox} = \text{Function}\left[b, \text{Evaluate}\left[\text{MiniMaxApproximation}\left[\text{GaussKummer}[b], \{b, (0, 1), n, m\}\right][2, 11]\right]\right]\]

For example, the \([4, 3]\) minimax approximation,

\[\text{In[44]}:= \text{GKapprox}[4, 3][b]\]

\[\text{Out[44]}= \frac{-0.081183 b^4 + 0.273498 b^3 + 1.77163 b^2 - 5.0554 b + 3.14159}{-0.14146 b^3 + 1.01321 b^2 - 1.8592 b + 1}\]

has (absolute) relative error at most \(2.3 \times 10^{-7}\), but is not “extreme perfect”.

\[\text{In[45]}:= \text{Plot}\left[10^7 \left(1 - \frac{\text{GKapprox}[4, 3][b]}{\text{GaussKummer}[b]}\right), \{b, 0, 1\}\right]\]

\[\text{Out[45]}= \text{Plot} \text{Graph}\]
Using the linear approximant $4h + \pi (1 - b)$ and noting that $b (1 - b)$ vanishes at both $b = 0$ and $b = 1$ leads to an optimal $[N + 2, M]$ extreme perfect approximant of the form

$$\pi_2 F_1 \left( -\frac{1}{2}, -\frac{1}{2}; 1; b \right) \approx 4b + \pi (1 - b) + \alpha b (1 - b) - \sum_{i=1}^{N} (b - p_i) \frac{\prod_{i=1}^{N} (b - p_i)}{\prod_{j=1}^{M} (b - q_j)},$$

where the parameters $\alpha$, $\{p_i\}_{i=1,\ldots,N}$, and $\{q_j\}_{j=1,\ldots,M}$ need to be determined. Implementation of the approximant is immediate.

In[46]:= EllipseApproximant[\alpha_, p_List, q_List] :=
  Function[b, Evaluate[b \alpha \text{Times}[b - p](1 - b) + \pi (1 - b) + 4b]]

After uniformly sampling the Gauss-Kummer function,

In[47]:= {xdata, ydata} = Table[{b, GaussKummer[b]}, {b, 0, 1, 0.001}];

we can use NMinimize and the $\infty$-norm to obtain the accurate approximants. For example, the (almost) optimal $[3, 2]$ approximant is computed using

In[48]:= NMinimize[\|ydata - EllipseApproximant[\alpha, \{p\}, \{q, r\}]\|_{\infty},
  \begin{pmatrix}
  \alpha & 0.22^\circ & 0.24^\circ \\
  p & 1.25^\circ & 1.35^\circ \\
  q & 3.4^\circ & 3.5^\circ \\
  r & 1.15^\circ & 1.25^\circ
  \end{pmatrix}]

Out[48]= \{0.0000206279, \{p \to 1.30685, q \to 3.46703, r \to 1.2114, \alpha \to 0.233508\}\}

leading to

In[49]:= EllipseApproximant[\alpha, \{p\}, \{q, r\}] \{b\}] \{%/\}

Out[49]= \frac{0.233508 (b - 1.30685) b (1 - b)}{(b - 3.46703) (b - 1.2114)} + \pi (1 - b) + 4b

This simple approximant has (absolute) relative error less than $6.5 \times 10^{-6}$. 

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\[ \text{In}[50]= \quad \text{Plot}\left[10^6 \left(1 - \frac{\%}{\text{GaussKummer}[b]}\right),\{b, 0, 1\}\right] \]

\[ \text{Out}[50]= \begin{array}{c}
\begin{array}{cccccc}
0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
6 & 4 & 2 & 0 & -2
\end{array}
\end{array} \]

\section*{Conclusions}

Computing the perimeter of an ellipse using a simple set of approximants demonstrates that \textit{Mathematica} is an ideal tool for developing accurate approximants to a special function. In particular:

- All special functions of mathematical physics are built in and can be evaluated to arbitrary precision for general complex parameters and variables.
- Standard analytical methods—such as symbolic integration, summation, series and asymptotic expansions, and polynomial interpolation—are available.
- Properties of special functions—such as identities and transformations—are available at \textit{MathWorld} \cite{8} and The Wolfram Functions Site \cite{9} and, because these properties are expressed in \textit{Mathematica} syntax, they can be used directly.
- Relevant built-in numerical methods include rational polynomial approximants, minimax methods, and numerical optimization for arbitrary norms.
- Visualization of approximants can be used to estimate the quality of approximants.
- Combining these approaches is straightforward and naturally leads to optimal approximants.

\section*{References}

\cite{1} S. Sykora. “Approximations of Ellipse Perimeters and of the Complete Elliptic Integral E(x).” (Aug 8, 2007) www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html.

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About the Author

Paul Abbott
School of Physics, M013
The University of Western Australia
35 Stirling Highway
Crawley WA 6009, Australia
tmj@physics.uwa.edu.au
physics.uwa.edu.au/~paul