Polar Zonohedra

We explore a "polar zonohedron function," which creates a wide variety of polyhedra bounded by rhombs and parallelograms, and obtain further variations using existing and new *Mathematica* functions.

Russell Towle

Polar zonohedra are convex solids bounded by $n(n-1)$ rhombic faces; twice this number of edges; and two plus this number of vertices (Figure 1). They are of interest not only by virtue of their beauty, but because, under certain conditions, they form isometric, orthogonal, solid "shadows" of hypercubes, cast into three dimensions from higher space [Coxeter 1973, chap. 2, 13]. That is, the edges of the hypercube are *equally* foreshortened under projection, and its square faces foreshorten into rhombs. We shall consider the code necessary to render these solids and create a number of attractive variations upon them.

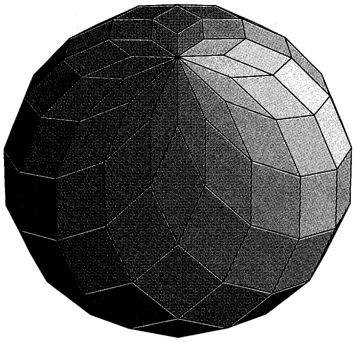

FIGURE 1. Polar zonohedron: $n = 12$, $p = 35.2643$.

A *zonohedron* is a polyhedron bounded by centrally symmetrical, parallel-sided, polygons: every edge of the polygon has an equal and opposite counter-edge, and so also the vertices have their diametrical opposites. The regular $2m$-gons ($m \geq 2$) and rhombs fit this criterion, although such a polygon need not be equilateral.

A zonohedron has itself central symmetry: every face and every edge has a "twin," similarly situated and parallel to it on the opposite side of the zonohedron. Since every face is a parallel-sided $2m$-gon, the edges occur in parallel sets. Fixing our attention upon any one such set, we observe a "zone" of faces girdling the zonohedron, each face in a zone having two sides belonging to the set. Such a set of parallel edges may be arbitrarily lengthened or shortened without affecting the interior angles of the faces or the dihedral angles of the zonohedron; a kind of prism ensues (Figure 2), in fact, the right regular $2m$-gonal prisms are themselves zonohedra, as is one of the five Platonic solids, the cube.

Several of the Archimedean solids are zonohedra (Figures 3, 4, 5). Any set of vectors in three dimensions may be said to *determine* a zonohedron (provided that not all the vectors lie in the same plane). The more symmetrically disposed these vectors are, the more symmetrical the determined zonohedron. The vectors from the centers to the vertices of the Platonic solids determine the most symmetrical of all zonohedra, which are without exception (like polar zonohedra) isometric solid shadows of hypercubes. The cube and tetrahedron determine Kepler's rhombic dodecahedron; the octahedron determines a cube; the icosahedron determines Kepler's rhombic triacontahedron; and the pentagonal dodecahedron determines a lovely *enneacontahedron*, bounded by thirty rhombs of one shape and sixty of another.

The *polar* zonohedra were first described by Russian crystallographer E.S. Fedorov about 100 years ago, although two, the cube and Kepler's rhombic dodecahedron, occur in nature as crystals and were known to the ancients. Like right regular pyramids, the polar zonohedra may be constructed

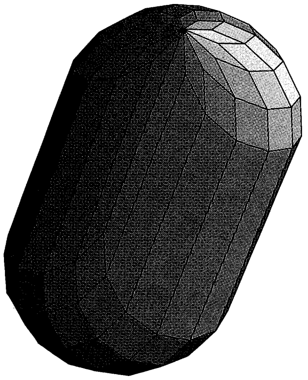

FIGURE 2. $n = 12$, $p = 35.2643$, factor = 12, ratio = 5.

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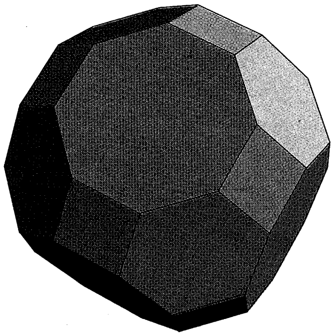

FIGURE 3. Truncated cuboctahedron.

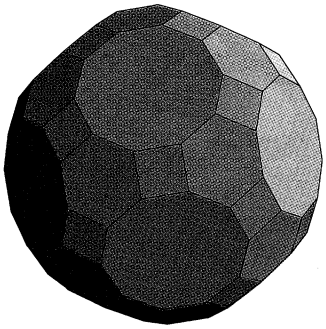

FIGURE 4. Truncated icosidodecahedron.

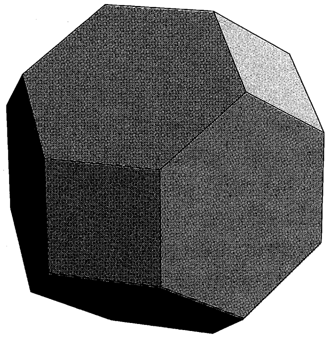

FIGURE 5. Truncated octahedron.

for any $n \geq 3$, and just as any such pyramid is determined by n evenly spaced generators of a right cone, those vectors also determine a polar zonohedron. As n tends towards infinity, the pyramid more and more closely approximates a cone, and the polar zonohedron approximates a solid-of-revolution of a sine curve [Coxeter and Chilton 1963]. Every pair of the n vectors corresponds to a rhombic face, and when a polar zonohedron forms a hypercubic shadow, the n directions in which edges occur correspond to the n dimensions of the hypercube.

At two special vertices, connected by an axis of cyclic symmetry, n edges meet: these are the "poles" of a polar zonohedron. Around each pole are n equal rhombs, like the petals of a daisy; beyond these, another set or "cycle" of n rhombs, and so on until the opposite pole is reached. There are $n - 1$ cycles of n rhombs, altogether.

It is most natural to imagine the axis of symmetry to be vertical, and identify it with the z axis in Cartesian coordinates. Having declared that evenly spaced generators of a right cone determine a polar zonohedron, it remains to characterize these vectors. Let the base of the cone be of unit circum-radius, the apex at the origin, and let the axis of the cone be the z axis. Then the x and y coordinates of the vectors will be those of a regular polygon, of the form $x = \cos 2\pi i/n$, $y = \sin 2\pi i/n$, $i = 0, 1, 2, \ldots, n - 1$. We could assign a height h for the z coordinate, but I prefer to use an angular argument: adopting the carpenter's term for slope of a rafter, I call the angle between the cone's base and its generators the *pitch*, p, and allow it to assume any measure between 0 and 90 degrees. Under these conditions, the cone (and the polar zonohedron determined by its generators) has zero volume when $p = 0$, and infinite volume when $p = 90$; for the height of the cone is simply the tangent of the pitch, and the "polar diameter" of the polar zonohedron is n times that tangent. The *projected* length of the vectors (upon the plane of the base) is unity, and their actual length is just the secant of the pitch.

A more interesting situation, with regard to volume, and more workable, with regard to the range of height across the domain of the pitch, arises when the generators of the cone (or the edges of the polar zonohedron) are assigned a fixed length – let us adopt unit magnitude. To achieve this, we multiply all the coordinates by the cosine of the pitch. Now the vectors (generators) behave like ribs of an umbrella as we vary the pitch, and have coordinates

$$(\cos p \cos 2\pi i/n, \cos p \sin 2\pi i/n, \sin p), \; i = 0, 1, 2, \ldots, n - 1$$

Here, *all* the coordinates respond to the "pitch transformation," the circum-radius of the base changes along with height, and most significantly, the volume function, of cone, pyramid, and polar zonohedron alike, now has *two* zeros, one at $p = 0$, the other at $p = 90$. The volume attains an absolute maximum when $p = \tan^{-1}\sqrt{1/2} \approx 35.2643$ degrees. The significance of this maximum has many aspects which I cannot pause to examine here, except to remark that *only* at this pitch does a polar zonohedron constitute a solid shadow of an n-dimensional cube.

The function `PolarZonohedron` starts by constructing a set of vectors of the form above. Then, all the vertices of a polar zonohedron are derived by taking sums of these, somewhat after the fashion of "completing the parallelogram." In this case, we "complete the polar zonohedron" by adding the n vectors, using the functions `FoldList` and `RotateLeft`. Then the vertices are translated so that the zonohedron is centered upon the origin. Finally, the list of polygons is obtained, using a two-dimensional `Table` to construct the $n - 1$ cycles of n rhombs.

Along with n and p, I have added two arguments to the `PolarZonohedron` function, to allow various sets of the n vectors to be varied in magnitude. These are denoted as `factor` and `ratio`: every `factor`-th vector is multiplied by `ratio`. If `factor` evenly divides n, then a fairly symmetrical zonohedron is determined (Figures 6, 7). When `factor` is set to n, only one of the vectors is multiplied by `ratio`, and one zone of rhombic faces is stretched into a zone of parallelograms. When `ratio` is

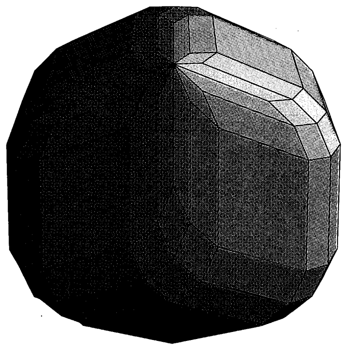

FIGURE 6. Polar zonohedron: $n = 12$, $p = 35.2643$, factor = 3, ratio = 3.

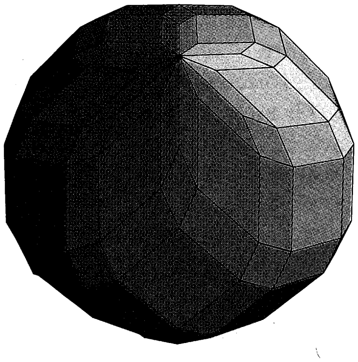

FIGURE 7. $n = 12$, $p = 35.2643$, factor = 2, ratio = 2.

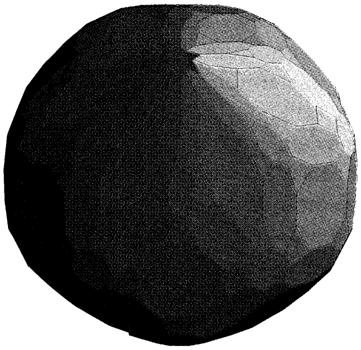

FIGURE 8. The zonohedron of Figure 7 truncated.

large enough (4 or 5 or more), a kind of "zonal prism" is developed.

Mathematica provides a number of functions that operate upon polyhedra generally, including `RotateShape`, `TranslateShape`, and `AffineShape` (found in the standard package ``Graphics`Shapes``), and `Truncate` (Figure 8), `OpenTruncate`, and `Stellate` (found in the package ``Graphics`Polyhedra``). Quite attractive effects are obtained by applying these latter to polar zonohedra. Inspired by the wonderful `HollowTriangle` function in Scott Kim's `Five Tetra` sample notebook (distributed with *Mathematica* Version 2.2), I devised a `HollowPolygon` function, which replaces each face with a "hollow" polygon (Figures 9, 10), and a `TruncateStellate` function, which replaces each face with a truncated pyramid (Figure 11). Both of these were based closely upon the existing `Stellate` function and are not confined to polyhedra with rhombic faces. The `HollowPolygon` function is a particular favorite of mine, for it seems to strike a pleasing balance between wireframe and shaded-polygon renderings of polyhedra.

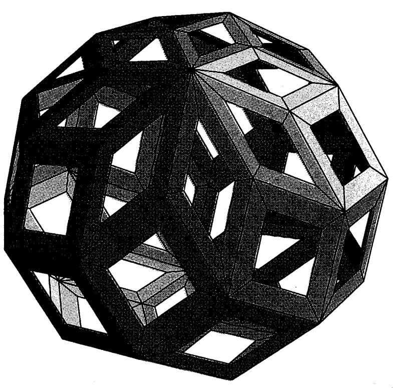

FIGURE 9. Polar zonohedron, $n = 7$, $p = 35.2643$, with hollow faces.

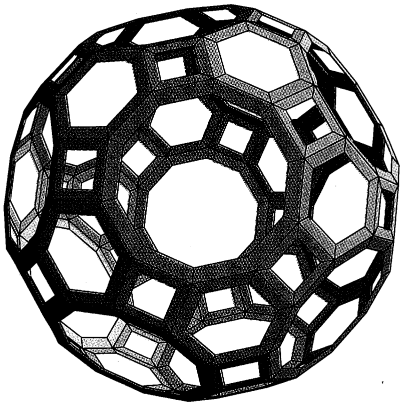

FIGURE 10. Truncated icosidodecahedron with hollow faces.

FIGURE 11. Truncated stellation of the polar zonohedron $n = 12$, $p = 35.2643$, factor = 3, ratio = 2.

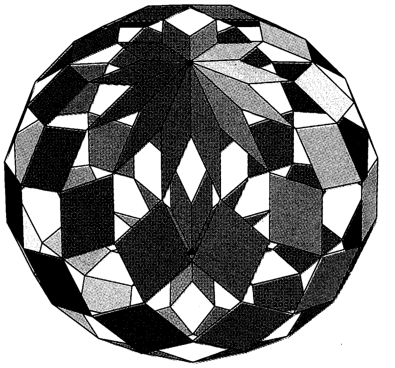

FIGURE 12. Alternate faces of the zonohedron of Figure 1.

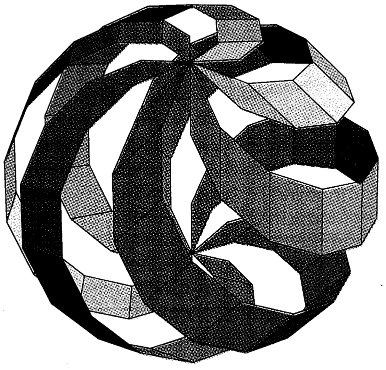

FIGURE 13. Alternate zones of the zonohedron of Figure 1.

```
PolarZonohedron[n_Integer, pitch_, factor_Integer, ratio_]:=
Block[{e, q, offset, verts},
    (*establish the n vectors*)
    q = Table[If[Mod[i, factor]==0, e=ratio, e=1];
          N[{e Cos[Degree pitch] Cos[2Pi i/n],
             e Cos[Degree pitch] Sin[2Pi i/n],
             -e Sin[Degree pitch]}], {i, 0, n-1}];
    (*take sums of the vectors cyclically,
                    transpose list into n+1 sets of n*)
    verts = Transpose[Table[
        FoldList[Plus, {0, 0, 0}, RotateLeft[q, i]],
        {i, 0, n-1}]];
    (*find distance between poles, center vertices upon ori-
gin*)
    offset = Abs[verts[[n+1, 1 ,3]]/2];
    Do[verts[[i, j, 3]] += offset, {i, n+1}, {j, n}];
    (*create the rhombic faces*)
    Graphics3D[
        Table[Polygon[
            {verts[[1+Mod[j, n+1], 1+Mod[k+1, n]]],
             verts[[1+Mod[j+1, n+1], 1+Mod[k, n]]],
             verts[[1+Mod[j+2, n+1], 1+Mod[k, n]]],
             verts[[1+Mod[j+1, n+1], 1+Mod[k+1, n]]]}],
            {j, 0, n-2}, {k, 0, n-1}]
    ]
] /; n > 2
    (*set defaults*)
PolarZonohedron[n_:12, pitch_:35.2643, factor_:1, ratio_:1]:=
    PolarZonohedron[n, pitch, factor, ratio]
```

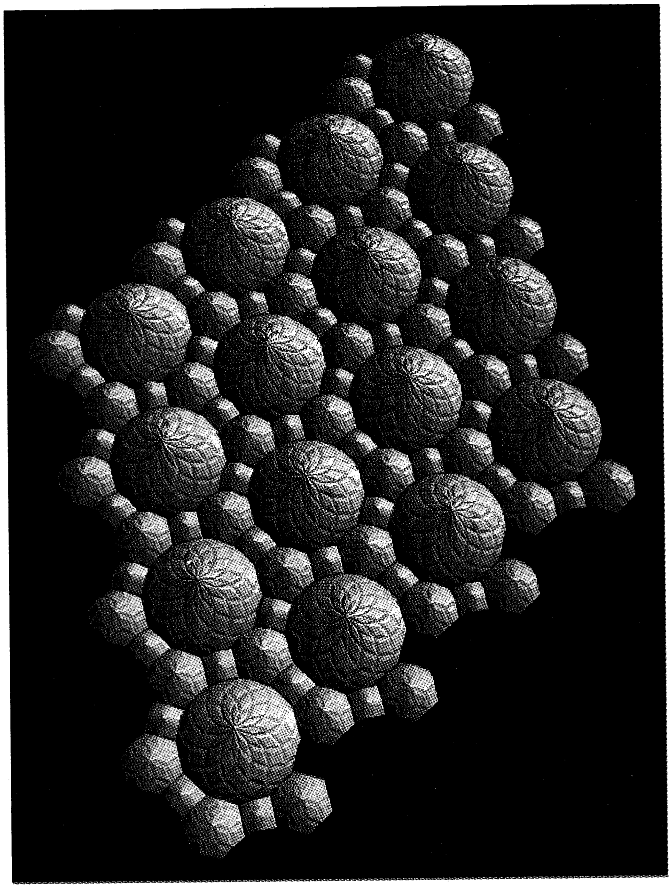

FIGURE 14. Truncated stellations of the northern hemispheres of polar zonohedra, inscribed within the tessellation of squares, hexagons, and dodecagons.

Simple modifications to the `PolarZonohedron` function itself can effect the following: overall scaling, independent of `ratio`, rotation about the z axis, or deletion of portions of the polygon list. For example, within the `Table` that constructs the polygons, one may limit the iterators to draw only, say, the "northern" hemisphere, or step through the iterators, to draw only every other face (Figure 12), or every other "half-zone" of faces (Figure 13). Again, by modifying the `Table` that establishes the vectors, a myriad of schemes to vary their magnitudes may be implemented. For instance, one may make magnitude depend upon the iterator itself, so that the n vectors have magnitudes 1, 2, ..., n..

When n is even, a polar zonohedron has a special cycle of faces, inscriptible in a regular n-gon, which forms the equator. It thus becomes possible to inscribe such polar zonohedra in tessellations of polygons; examples include the tessellation of octagons and squares, or dodecagons, hexagons and squares (Figure 14). When $n = 4$ and $p = \tan^{-1}\sqrt{1/2} \approx 35.2643$ degrees, we obtain Kepler's rhombic dodecahedron, which may be inscribed in the tessellation of squares. This zonohedron also close-packs to fill space (Figure 15) and is employed by bees and kindred creatures in their honeycombs: what appear to be hexagonal cells are actually rhombic dodecahedra, which under one construction may be taken as the solid of intersection of four hexagonal prisms. These plane and solid tessellations may be created using the function `TranslateShape`.

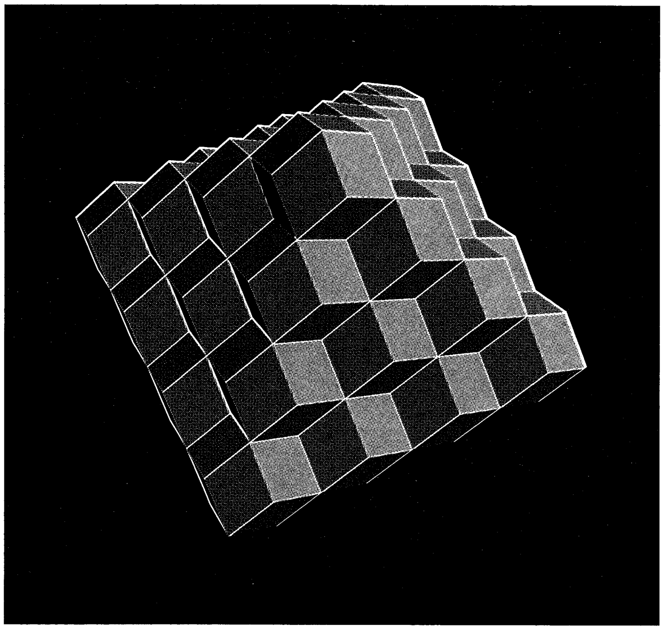

FIGURE 15. An octahedral cluster of the close-packing of rhombic dodecahedra.

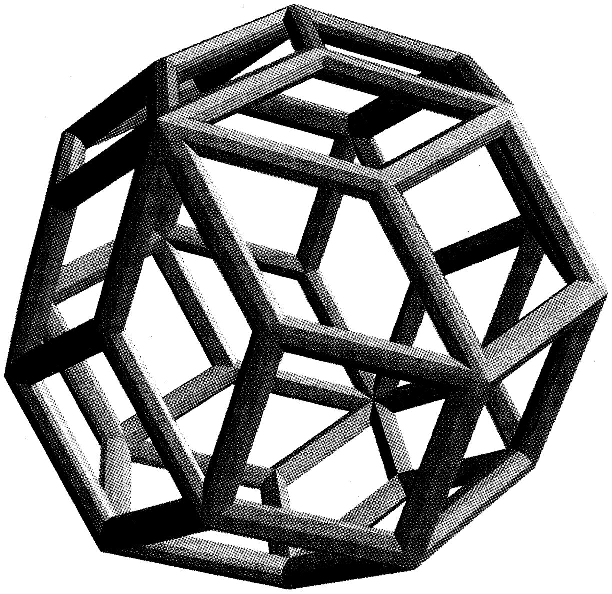

FIGURE 16. An $n = 6$, $p = 35.2643$ polar zonohedron whose edges have been replaced by irregular prisms.

References

Coxeter, H.S.M. 1973. *Regular Polytopes*. Dover.

Coxeter, H.S.M., and W.W.R. Ball. 1987. *Mathematical Recreations and Essays*. Dover. (See pp. 141–144).

Coxeter, H.S.M, and B.L. Chilton. 1963. *American Mathematical Monthly* 70:946–951.

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The electronic supplement contains the notebook `Cyclic Polyhedra`.