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Clipping Polygons

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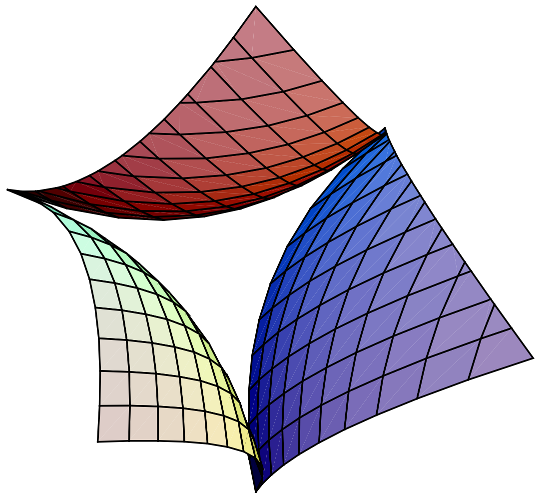
A routine for clipping polygons has a variety of graphics applications.

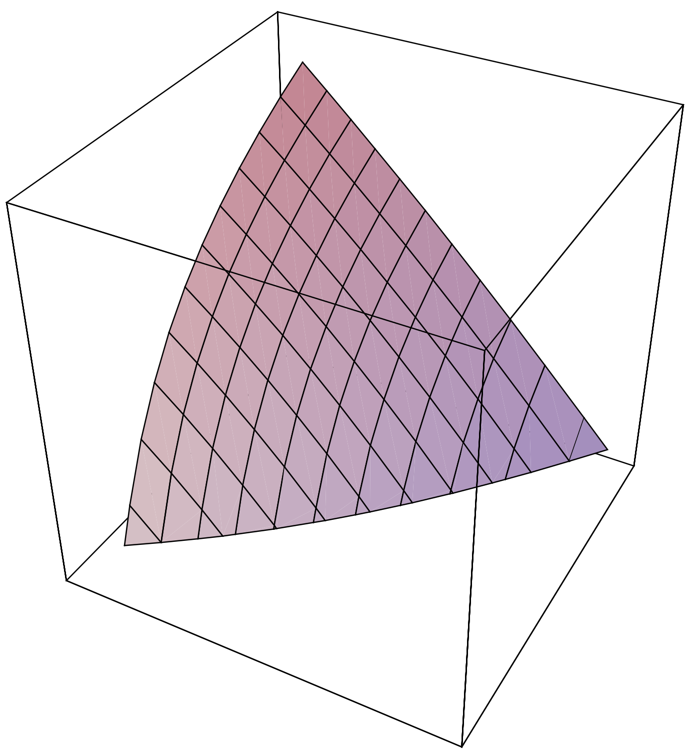
■ Introduction

The standard *Mathematica* function **`ParametricPlot3D`** requires the parameter domain to be a rectangle. Suggestions for handling nonrectangular domains usually involve triangulation of the domain and a version of the function **`TriangularSurfacePlot`** from the **`ComputationalGeometry`** package (see [1] and [5]). The package **`SurfaceClip.m`** takes a different approach. A rectangle containing the parameter domain is first paved with a set of rectangles, which are then clipped to the shape of the domain. The parametrization functions are then applied to the vertices of the clipped polygons.

As an example, Figure 1 shows a portion of a complicated surface called the bisectrix of the tetrahedron (see [2], [3], [4]). The three pieces of the figure are congruent to the piece of surface shown in Figure 2.

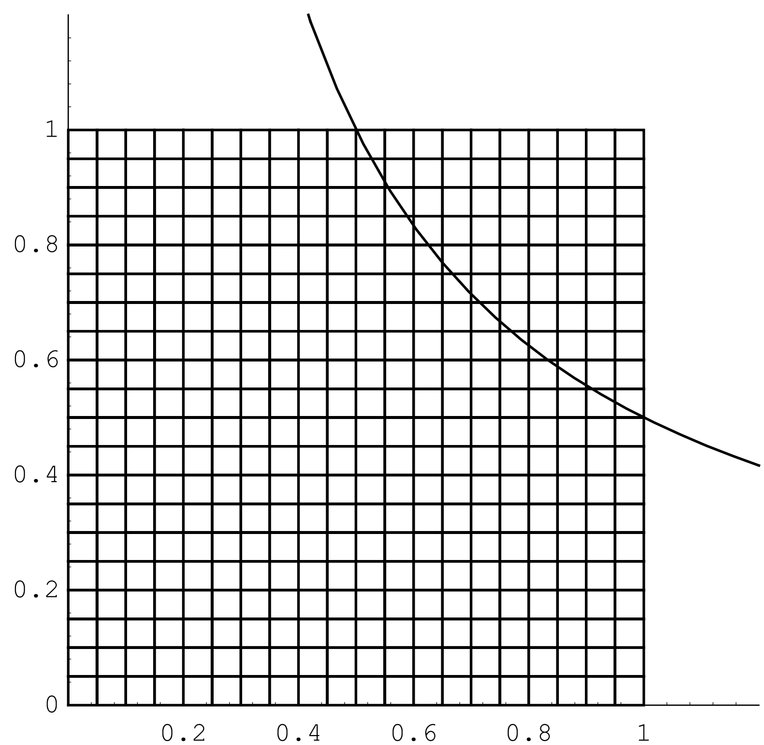


▲ **Figure 1.** Three surface patches.

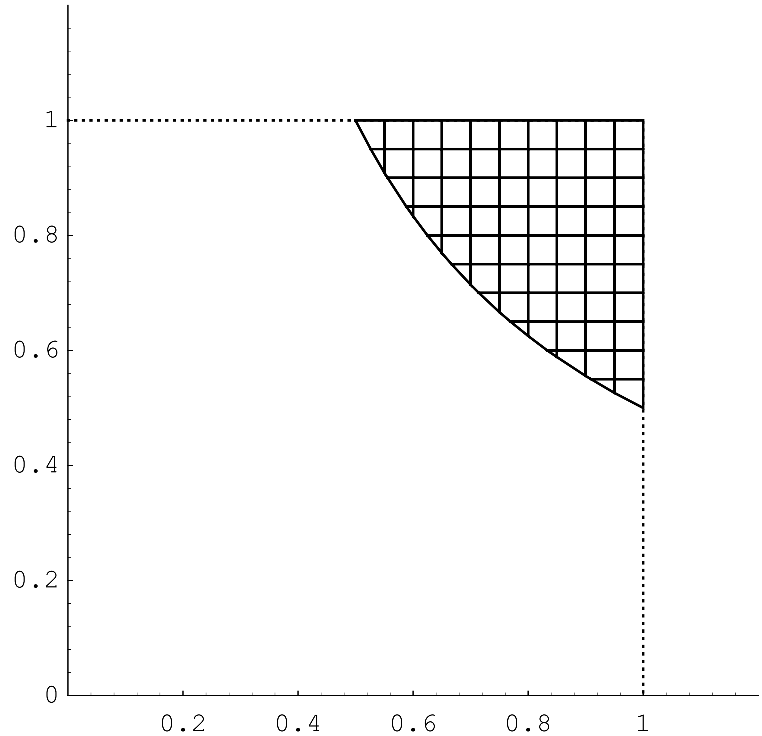


▲ **Figure 2.** A single patch.

The single patch has a reasonably simple parametrization ($x = -3 + 2\,p + 1/p\,q$, $y = 3 - 2\,q + 1/p\,q$, $z = -3 + 2\,p + 2\,q$), but to plot the edges correctly the parameters must be restricted to the portion of the unit square in the *p*-*q* plane lying above the hyperbola $2\,p\,q = 1$ (Figure 3). To draw Figure 2, a grid of rectangle polygons covering the square was first defined (Figure 3). This polygonal grid was then clipped using the expression $1 - 2\,p\,q$, giving the set of polygons shown in Figure 4. Clipping is accomplished by discarding polygonal vertices lying in the region where $1 - 2\,p\,q$ is positive. Finally, the parametric equations were applied, mapping each remaining polygon to a three-dimensional polygon. (The three pieces of Figure 1 are images of the piece of Figure 2. They were computed using homogeneous transformations from the Wickham-Jones packages [5].)



▲ **Figure 3.** Paving a rectangle.

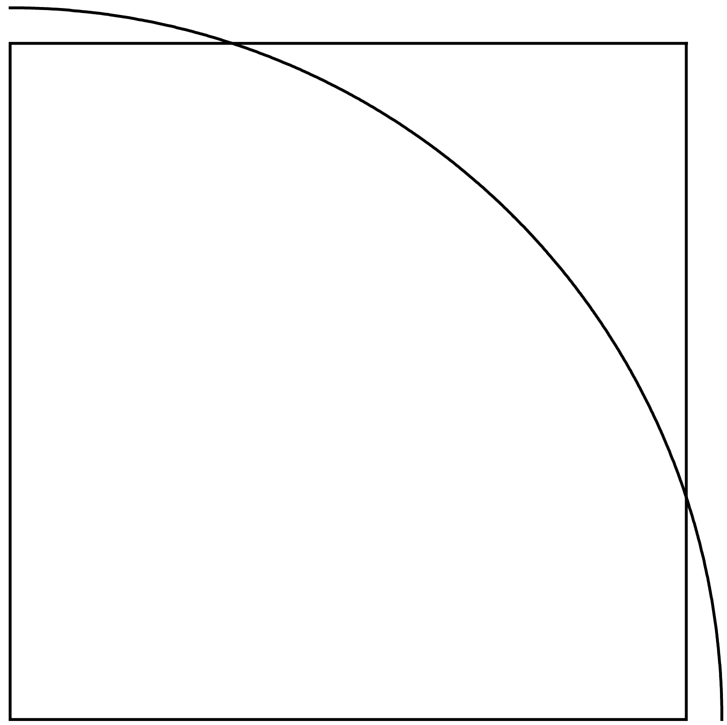


▲ **Figure 4.** Clipped paving.

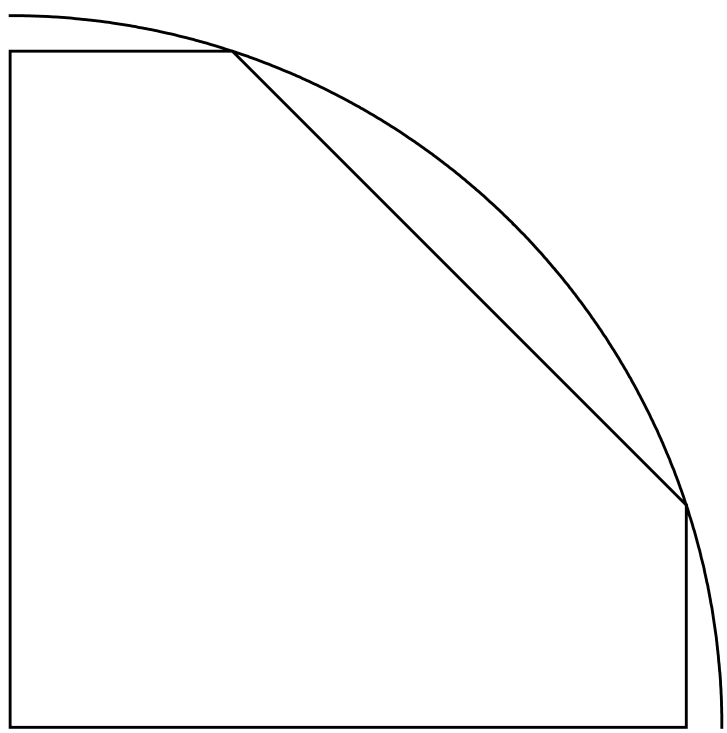
The package **SurfaceClip.m** contains the function **ClippedParametricPlot3D** that automates this procedure. The function takes four arguments and the syntax is that of the built-in function **ParametricPlot3D** with an extra argument added. The first three arguments of the two functions are the same, defining the parametrization functions and a rectangle in the parameter plane. The last argument is a list of clipping functions that are applied in succession.

■ Clipping a Polygon

The basic routine of the package is a function that clips a polygon along a curve given by an implicit equation $f(x, y) = 0$. The coordinates of a vertex are substituted into f. Those vertices for which the result is positive are discarded and those for which the result of the substitution is negative or zero are kept. Figures 5 and 6 illustrate the process for a square clipped by the circle $f(x, y) = x^2 + y^2 - 1 = 0$.



▲ **Figure 5.** Circle and square.



▲ **Figure 6.** Clipped square.

In Figure 5 a square polygon, shown in outline, is crossed by the circle, only a portion of which is shown. In Figure 6 the polygon has been clipped by discarding the vertex that lies outside the circle and adding vertices where the circle crosses the sides of the polygon. The clipped polygon lies entirely within or on the clipping circle.

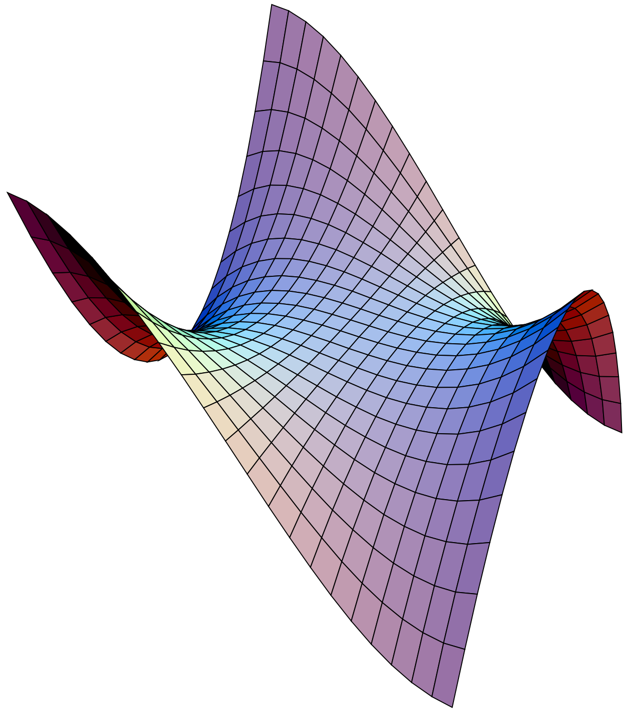
The new vertices are computed as follows. Suppose that p and q are adjacent vertices on which the clipping function f has different signs. The edge joining p and q may be parametrized as $l(t) = (1 - t)\, p + t\, q$, and then $g(t) = f(l(t))$ is a real-valued function of the real variable t, which has at least one root between 0 and 1. A root is located with **`FindRoot`** and the value substituted into $l(t)$ to define the new vertex.

Since vertex coordinates are converted to approximate real numbers for this computation, problems with roundoff may occasionally arise. For this reason the test for retaining a vertex is not $g(t) \leq 0$ but $g(t) \leq \epsilon$ where ϵ is small number internally referred to as **`Fuzz`**. The default value of **`Fuzz`** is 10^{-5} but may be changed by the user.

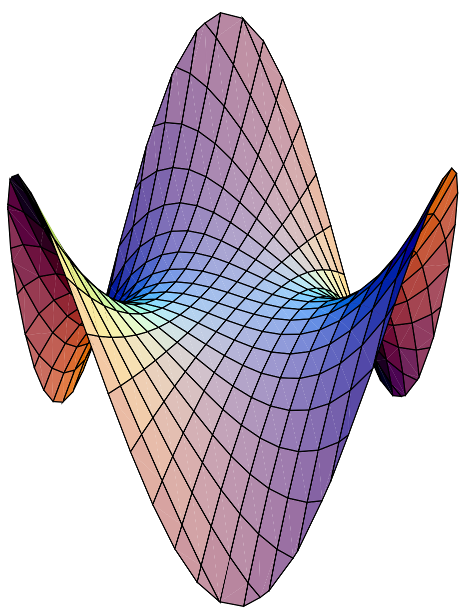
■ Clipping with Surfaces

Nothing in the above description of the clipping process restricts the polygons to two dimensions or the clipping function to two variables. The process may be used to clip one surface by another. This extends a function in the Wickham-Jones package that clips a surface with a plane. Here are three examples.

Figure 7 shows the Monkey Saddle, $z = x^3 - x\, y^2$, plotted over the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. The three-fold symmetry of the surface may be emphasized by clipping it to the inside of the cylinder $x^2 + y^2 = 1$. The clipped surface is shown in Figure 8.

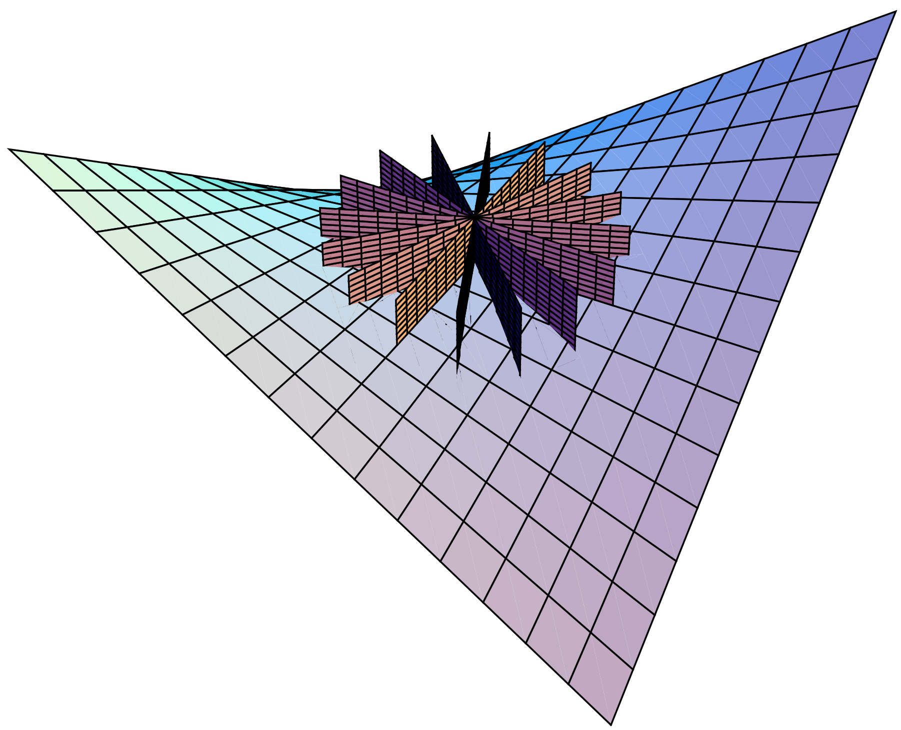


▲ **Figure 7.** $z = x^3 - x y^2$.

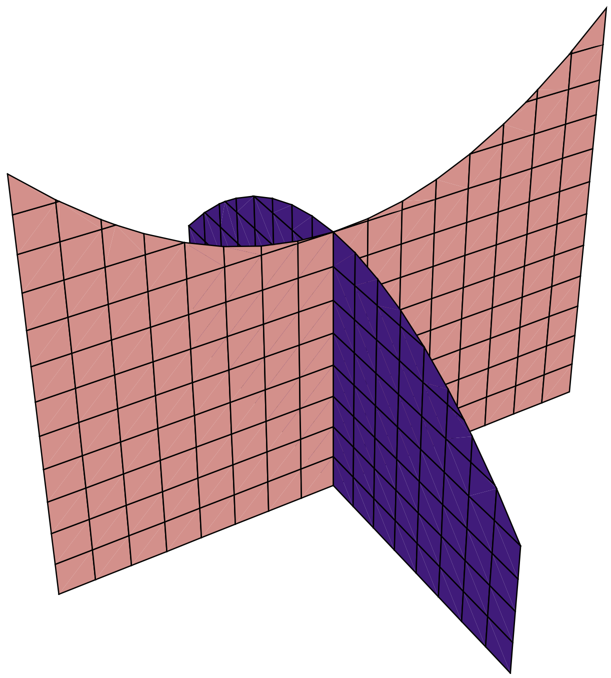


▲ **Figure 8.** Clipped by $x^2 + y^2 - 1$.

Clipping a plane by a surface gives a section of the surface. An application of this idea is shown in Figures 9 and 10. The Gaussian and mean curvatures of a surface at a point may be defined in terms of the curvature of the curve of intersection of the surface with a plane through the point and containing the surface normal. Figure 9 shows a surface ($z = x\,y$) and a sampling of planes containing the normal to the surface at a fixed point. Clipping the planes by the surface reveals the normal sections. Figure 10 shows the sections with maximal and minimal (signed) curvature. The package contains a utility, **`Grid`**, to generate planes to be clipped.

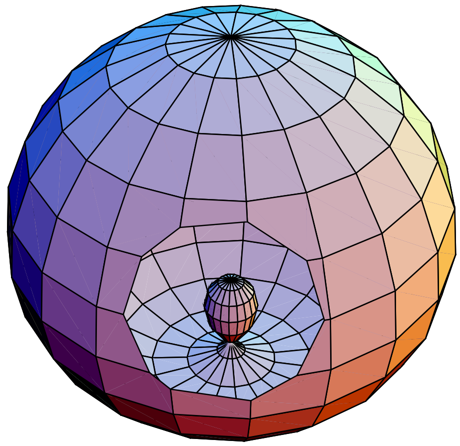


▲ **Figure 9.** Normal planes.

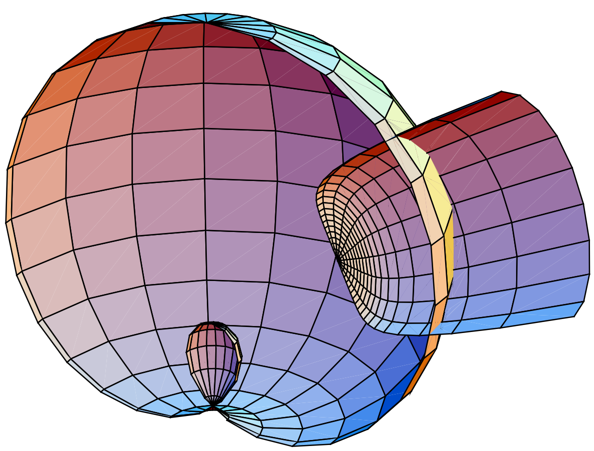


▲ **Figure 10.** Normal sections.

In Figure 11 the surface is generated by rotating a limaçon about its line of symmetry. A hole has been punched in the surface to reveal the inner lobe. Figure 12 indicates how the clipping was accomplished. The figure shows a cross section (clipped by a plane) of the rotated limaçon together with the surface used to punch the hole. The punch surface is also a surface of revolution. The profile curve is $z = x^8$ rotated about its axis of symmetry—yielding $z = (x^2 + y^2)^4$—and then moved into position by an appropriate geometric transformation.



▲ **Figure 11.** Rotated limaçon.



▲ **Figure 12.** Punch in position.

■ Conclusion

The package **`SurfaceClip.m`** contains the function **`ClipGraphics`** that clips one surface by another. **`ClipGraphics`** takes three arguments. The first argument is a **`Graphics`** object, a **`SurfaceGraphics`** object, a **`Graphics3D`** object, or a list of these. **`SurfaceGraphics`** objects are converted to **`Graphics3D`** objects before clipping. The second argument is a clipping function or list of clipping functions, and the third argument is a variable list. The variable list matches variables with coordinates. The variable list $\{p, q, r\}$ indicates that the variable p refers to the first coordinate, the variable q refers to the second, and the variable r to the third. In addition to polygons, **`ClipGraphics`** clips lines and points as well.

■ References

[1] P. Abbott, “Graphs over Convex Polygons,” *The Mathematica Journal*, **14**(2), 1994 pp. 21–22.

[2] W. A. Beyer and B. Swartz, “Bisectors of Triangles and Tetrahedra,” *The American Mathematical Monthly*, **100**, 1993 pp. 626–640.

[3] G. Gunther and J. B. Wilker, “The Bisectrix of a Tetrahedron,” *Mathematika*, **39**, 1992 pp. 93–103.

[4] G. Helzer, “The Bisectrix of a Tetrahedron; An Application of Symbolic Computation,” *Mathematica in Education and Research*, **6**(2), 1997 pp. 2–13.

[5] T. Wickham-Jones, *Mathematica Graphics*, New York: Springer-Verlag, 1994.