

Spatial Inversion: Reflective Anamorphograms

Philip W. Kuchel

The British flag (or 'Union Jack') is transformed so that if it is viewed from above with a straight-sided reflective cone placed in its center, the viewer perceives its original form to lie in the plane at the base of the cone. This so-called anamorphic art is readily studied by using the versatile graphics functions of *Mathematica*.

■ Introduction: Inversion of Space and Optical Transforms

This article discusses the fascinating idea that you can take the entire region of a two-dimensional surface between the outer edge of a circle and infinity (in all directions) and compress it into the interior of the circle with a single mathematical transformation. *Circle inversion* is one way to do this, and an explanation of its properties, when applied to simple geometrical objects, constitutes an important branch of geometry [1]. However, classical planar circle inversion is merely a special case of more general inversion transformations that can be applied to geometrical objects around a circle. In turn, these planar transformations can be generalized to three (and more) dimensions. Inversion in a sphere can be usefully employed to generate new curvilinear coordinate systems from older or more familiar ones [2–5]. If the coordinate lines intersect at right angles, the transformed ones will also be orthogonal because inversions preserve angles. Some rather extraordinary coordinate systems and corresponding surfaces can be generated in this way [2, 3].

The curved mirrors in fun houses elicit reactions of delight through the unexpected transformations they bring about when forming the images of the beholder. Likewise, the direction signs on road surfaces have strange proportions when viewed directly from above, and yet when viewed obliquely from the driver's position in a car, they appear perfectly well proportioned. These are but two examples of what is generally called *anamorphic* art from the Greek (ana = again, morphe = form) [6]. The transformation in the first case is wrought by the wavy mirror and can be classified as a reflective transformation: the undistorted object is in front and the image is perceived by the viewer to be behind the mirror. In the second case, the transformation of the object is achieved by changing the

angle of its projection onto the imaging surface that in most cases is the retina of the eye, or the film (or charge-couple device) of a camera.

■ Circle Inversion

Circle inversion entails taking a point O in the plane and specifying it as the center of a circle C of radius a . A point P is transformed to the point P' such that [1]

$$\overline{OP} \cdot \overline{OP'} = a^2. \quad (1)$$

It is relatively simple to prove a range of theorems involving this transformation.

1. A circle that is completely outside C is transformed to a circle wholly inside C , but not passing through the O (the center of inversion), and vice versa.
2. A circle that intersects C and passes through O is transformed to a straight line that passes through the points of intersection of the two circles, and vice versa.
3. The inverse of a line that does not pass through C is a circle inside C that passes through O , and so forth [1].

■ Reflective Circle Inversion

Circle inversion can also be invoked optically by using a conoidal reflecting surface that was described several years ago [7] (Figure 1). The object lies in the plane outside the base of the conoid. When the conoid is viewed directly from above its apex, an image appears to lie in the plane within the confines of its base. The aforementioned theorems can be readily verified with this device. Since the transformation is so regular, the transforms of various objects or anamorphograms can be made by using a ruler and compass. Specifically, straight lines in the image are made from arcs of circles that are drawn with the compass in the object field. However, this geometrical construction is tedious, and it is essentially redundant now that it can be performed so readily using *Mathematica*.

In the next section, we show how to take the reflective inversion of the Union Jack in a very direct manner. In order that the reflective surface can be easily constructed, rather than having to turn a circle inverting conoid from metal on a lathe [7], we have used the transformation that describes the reflection invoked by a straight-sided cone. Such cones can be made from discs of shim brass or aluminum-coated Mylar sheeting, or turned from solid metal if a lathe is accessible.

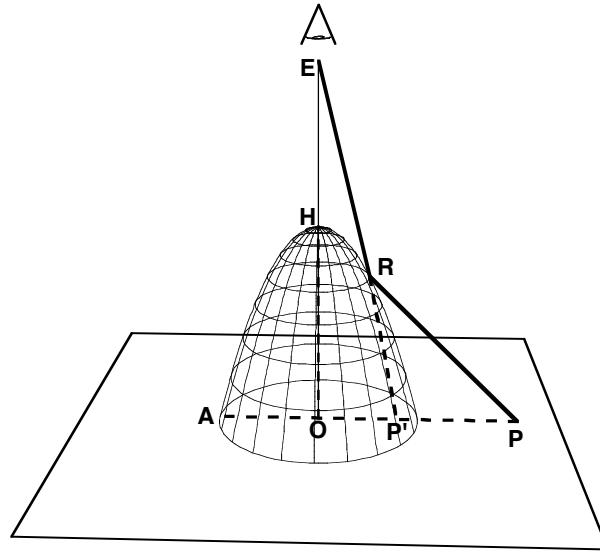


Figure 1. The arrangement of a circle-inverting reflective conoidal anamorphoscope [7]. The object field is a Euclidean plane that contains the anamorphogram, or picture, which is transformed into a familiar image that is perceived to lie within the confines of the base of the conoid (image field) when it sits on the plane and is viewed directly from its apex. The height of the conoid is h , the radius of the base is a , and the inverse of the point P is P' , such that it conforms to equation (1). The angle of incidence of the light from P' at the surface of the conoid R is equal to its angle of reflection up to the eye at E .

■ Reflective Inversion in a Simple Cone

Suppose that we use the general setup shown in Figure 1, but instead employ a straight-sided cone [7]. The path of a ray of light from the object field to the eye is indicated by the dashed line, while the perceived origin of the light is at the point of intersection of the thinner dashed line and the base plane (point P). For simplicity, it is assumed that the point of observation is at infinity above the plane of the object; hence, the light ray is specified as being parallel to the axis of symmetry of the cone. This is a mild approximation to having a viewing height of ≈ 20 times the height of the cone. The transformation rule is derived by considering Figure 2; here are the only physical rules to be used in arriving at the transformation formulas.

1. The angle of incidence of the light ray at the surface of the cone is equal to the angle of reflection.
2. The perceived source of a light ray lies at the intersection of the straight line drawn from the eye, intersecting the cone at the point of reflection and then intersecting the base plane.

3. When the solid angle at the vertex of the cone is 90° , the light from infinity is perceived to originate at the center of inversion. If the solid angle is less than 90° , then the horizon circle of the object field is not at infinity.

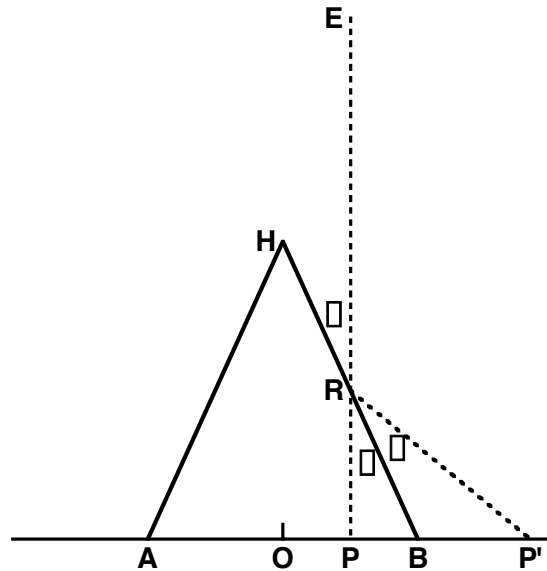


Figure 2. The central longitudinal section of a straight-sided cone showing the construction lines and light rays that are necessary to develop the mathematical expressions for the reflective inversion that is generated by such a cone. The symbols on the diagram correspond to those in Figure 1. The mathematical expressions are given in equations (2) to (5) for which $a = \overline{OB}$ and $h = \overline{OH}$. In addition, α is the angle of incidence and reflection of the light ray, while, in general, for a straight-sided cone $\overline{OP} \cdot \overline{OP'} \neq a^2$.

Therefore, to construct an anamorphogram, we begin with the image that will be produced. It is composed of line segments joining (x, y) coordinate points. The transformation of these to give the new points, in what will be the object field, is given after some trigonometric analysis by

$$x \rightarrow x' = \frac{x(d_1(x^2 + y^2)^{1/2} + d_2)}{(x^2 + y^2)^{1/2}} \tag{2}$$

$$y \rightarrow y' = y \frac{x'}{x}, \tag{3}$$

where the so-called dilation factors are

$$d_1 = 1 - \frac{bt}{r} \tag{4}$$

$$d_2 = bt, \tag{5}$$

where b is the height of the cone, r is the radius of its base, and

$$t = \text{Tan}[2 \text{ArcTan}[r/b]]. \quad (6)$$

Let us now consider a familiar emblem that has a particularly intriguing transform.

■ Inverting the Union Jack

The British flag (or Union Jack) is especially striking because of its numerous angles, bright colors, and yet geometrically simple construction. First, the flag is defined geometrically by representing it as a set of triangles, parallelograms, and more general polygons, in which the sides are further divided into short linear segments. After the points are transformed, the sides of the resulting polygonal sections will be good approximations to smooth curves. Figure 3 shows such a representation of the Union Jack.

The Union Jack is not completely symmetric: it is intended to have a well-recognized top and bottom edge. In times of military or civilian emergency, it can be flown upside down to attract attention. Nevertheless it has two-fold rotational symmetry about its center so only two Cartesian quadrants need be represented as a list of coordinates, and the other two can be obtained by the appropriate rotation. Specifically, the upper-right quadrant is rotated through 180° to give the lower-left one, and the lower-right quadrant is rotated through 180° to give the upper-left one. The transformation that delivers this rotation through the angle α centered about the origin $(0, 0)$ is

$$x \rightarrow x' = x \text{Cos}[\alpha] + y \text{Sin}[\alpha] \quad (7)$$

$$y \rightarrow y' = x \text{Sin}[\alpha] + y \text{Cos}[\alpha], \quad (8)$$

and of course in the present case $\alpha = 180^\circ$.

The actual function used in the *Mathematica* program is called `rotator` and is implemented as follows.

```
rotator[points_] := Map[
  {{Cos[π], Sin[π], -Sin[π], Cos[π]}.# &,
  points];
```

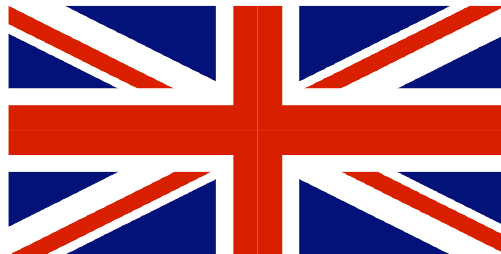


Figure 3. The Union Jack.

The inverted flag is produced by subjecting the List of coordinates to the transformations expressed in equations (2) to (5).

```

inverter[points_, h_, r_] := With[{
  t = Tan[2 ArcTan[r/h]],
  Map[{{#[[1]], #[[2]] / #[[1]]} ((1 - h t) Norm[#] + h t) / Norm[#] &,
  points}];

```

The inverted flag can be optically transformed back to the undistorted image of the Union Jack by placing a reflecting cone of the right height and radius at its center and viewing the whole setup from above the apex of the cone. Thus, we will have inverted the Union Jack mathematically and reinverted it optically (Figure 4).

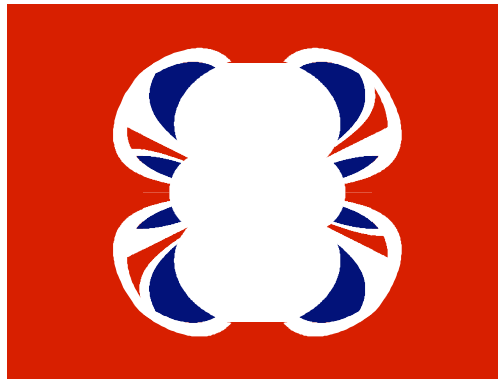


Figure 4. The reflective inversion of the Union Jack obtained by applying the coordinate transformation given by equations (2) to (5) to the polygonal (x, y) coordinates that define Figure 3. This figure is inverted back to an undistorted Union Jack by a straight-sided cone that has a base circle that touches the inside of the central four cusps of the pattern and is of a height that is $8/7$ of the diameter of the base; that is, the solid angle of the cone is 47.3° .

■ Conclusions

Mathematica provides an elegant way to set up geometrical objects and invoke reflective inversion transformations on them. In principle there is no limit to the implementation of reflective inversions of any reflective surface, even those that are not rotationally symmetrical.

The *Mathematica* notebook PolyFlag.nb performs the Union Jack transformation and the notebook DialInv.nb provides the more complicated example of inverting a clock face.

■ Acknowledgments

This paper was written while on a stimulating sabbatical leave with Professor Peter Stilbs at the Royal Institute of Technology-KTH in Stockholm. I would also like to thank Dr. Konstantin Momot for creating Figure 1 in *Mathematica*.

■ References

- [1] H. S. M. Coxeter, *Geometry*, Cambridge: Cambridge University Press, 1966.
- [2] P. Moon and D. Spencer, *Field Theory for Engineers*, Princeton: Van Nostrand, 1961.
- [3] P. Moon and D. Spencer, *Field Theory Handbook*, 2nd ed., Berlin: Springer-Verlag, 1967.
- [4] P. W. Kuchel, B. T. Bulliman, and E. D. Fackerell, "Bi-cyclide and Flat-ring Cyclide Coordinate Surfaces: Correction of Two Expressions," *Mathematics of Computation*, **49**, 1987 pp. 607–613.
- [5] P. W. Kuchel and B. T. Bulliman, "Expressions for Surfaces in Disk-cyclide Coordinates: Derivations using Symbolic Computation," *J. Franklin Inst.* **325**, 1988 pp. 505–508.
- [6] M. Gardner, "The Curious Magic of Anamorphic Art," *Scientific American*, **232**, 1975 pp. 110–116.
- [7] P. W. Kuchel, "Anamorphoscopes: A Visual Aid for Circle Inversion," *Math Gazette*, **63**, 1979 pp. 82–89.

■ Additional Material

PolyFlag.nb
DialInv.nb

Available at www.mathematica-journal.com.

About the Author

Philip W. Kuchel is McCaughey Professor of biochemistry at the University of Sydney, where he has been since 1980. He works in the general area of physical biochemistry and makes extensive use of nuclear magnetic resonance (NMR) spectroscopy to study biochemical reactions in cellular systems. His recent book, co-authored with Peter Mulquiney (*Modelling Metabolism with Mathematica*, (2003) CRC Press, Boca Raton), is an example of the extensive use of *Mathematica* in his research.

Philip W. Kuchel
School of Molecular and Microbial Biosciences,
University of Sydney, NSW 2006, Australia
p.kuchel@mmb.usyd.edu.au