Elliptic Rational Functions

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This article introduces a new algorithm for computing the elliptic rational function $R_n(\xi, x)$. The key feature of the algorithm is a symbolic computation of some of the $R_n(\xi, x)$ using only elementary functions. Our algorithm is more efficient than traditional methods based on the Jacobi elliptic functions and the complete elliptic integral, which can be numerically intensive. $R_n(\xi, x)$ is used extensively in signal processing and system design as the best minimax approximation of a unit square pulse.

■ Introduction

Designers of many practical systems searched for a rational function $R_n(\xi, x)$ in the variable x that has

- the equiripple property and $|R_n(\xi, x)| \leq 1$ over the interval $|x| \leq 1$
- the largest value of $\min(|R_n(\xi, x)|)$ for $|x| \geq \xi > 1$
- the minimal order n

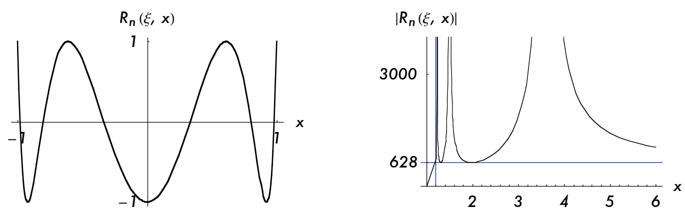
The rational function with those properties was found [1, 2] by using the Jacobi elliptic functions [3] and it is referred to as the *elliptic rational function* [4].

A function has the *equiripple* property if it oscillates between maximums and minimums of equal amplitude [1]. A quotient of two polynomials is called a *rational function* in the variable x, and the highest power in the polynomials is called the *order* of the rational function. The minimal value of $|R_n(\xi, x)|$ for $|x| \geq \xi$ is called the *discrimination factor* and is designated by $L_n(\xi)$. In signal processing theory ξ is known as the *selectivity factor* and can be any real number greater than 1, $\xi > 1$. Note that $R_n(\xi, x)$ is not a rational function in ξ. A typical plot of $R_n(\xi, x)$ is shown for $n = 6$ and $\xi = 1.2$.

We have defined the function `EllipticRationalFunction` that implements $R_n(\xi, x)$. The algorithm is detailed in the subsequent sections.

```
In[1]:= With[{n = 6, ξ = 1.2, R = EllipticRationalFunction[n, ξ, x]},
         L = DiscriminationFactor[n, ξ];
         a = Plot[EllipticRationalFunction[n, ξ, x], {x, -1, 1},
           PlotStyle → {Thickness[0.005]}, AxesLabel → {x, R},
           Ticks → {{-1, 1}, {-1, 1}}, DisplayFunction → Identity];
         b = Plot[Abs[EllipticRationalFunction[n, ξ, x]],
           {x, 1, 5 ξ}, PlotStyle → {Thickness[0.005]},
           AxesLabel → {x, Abs[R]}, PlotRange → {Automatic, {0, 4000}},
           Ticks → {Automatic, {Round[L], 3000}},
           GridLines → {{ξ}, {L}}, DisplayFunction → Identity];
         Show[GraphicsArray[{a, b}]]]
```

From In[1]:=



We used our symbolic algorithm for the elliptic rational function to optimize the symbolic performance of analog and digital systems. This optimization is not possible using traditional numeric algorithms. We derived closed-form formulas for designing high-speed low-consumption systems known as quadrature mirror filter banks [5]. $R_n(\xi, x)$ is extensively used in analog signal processing as the best approximation function [6].

We found a new function, known as *Minimum-Q Elliptic* [4, 7], by symbolically optimizing the elliptic rational function. Minimum-Q Elliptic became a standard function in manufacturing integrated filters [7]. In addition, again using symbolic optimization, we implemented a very efficient digital signal processing (DSP) system using programmable logic devices and very large-scale integrated circuits [5, 8]. By an efficient DSP system, we mean processing by multiplierless systems that consist of a small number of adders and binary shifters.

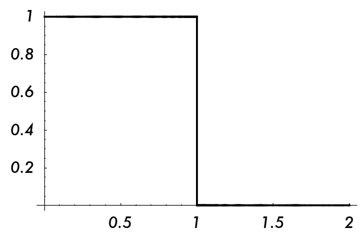
■ Application

The quintessence of the importance of $R_n(\xi, x)$ can be illustrated by the fact that $R_n(\xi, x)$ can be used for generating the best minimax approximation of a unit square pulse [1, 4]:

$$\frac{1}{1 + \epsilon^2 R_n^2(\xi, x)}. \tag{1}$$

```
In[2]:= With[{ϵ = 0.01, n = 32, ξ = 1.0001},
          Plot[1/(1 + ϵ^2 EllipticRationalFunction[n, ξ, x]^2),
           {x, 0, 2}, PlotStyle → {Thickness[0.007]}]]
```

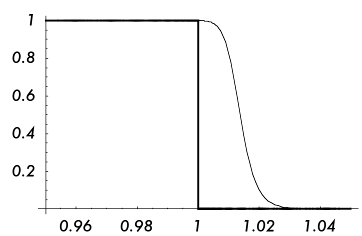
From In[2]:=



Chebyshev polynomials, for the same order, give a far inferior approximation (thin line) of the square pulse as shown in the following figure.

```
In[3]:= With[{ϵ = 0.01, n = 32, ξ = 1.0001},
          Plot[{1/(1 + ϵ^2 EllipticRationalFunction[n, ξ, x]^2),
            1/(1 + ϵ^2 ChebyshevT[n, x]^2)}, {x, 0.95, 1.05},
           PlotStyle → {Thickness[0.007], Thickness[0.002]},
           AxesOrigin → {0.95, 0}]]
```

From In[3]:=



Elliptic rational functions contain the one free parameter ξ that is used to adjust the slope of the pulse approximation. Chebyshev polynomials have no such parameter.

The pulse approximation with elliptic rational functions has many important applications in analog and digital signal processing and system design. Symbolic computation of elliptic rational functions and powerful symbolic algebra environments such as *Mathematica*, made many successful industrial designs possible [4, 5, 6, 7, 8]. These designs produced robust systems (such as analog and digital filters), shortened time to market, and helped designers make cost-effective solutions.

■ Definition

We define the elliptic rational function in terms of the Jacobi elliptic functions as

$$R_n(\xi, x) = \operatorname{cd}\left(n \frac{K\left(\frac{1}{L_n(\xi)}\right)}{K\left(\frac{1}{\xi}\right)} \operatorname{cd}^{-1}\left(x, \frac{1}{\xi}\right), \frac{1}{L_n(\xi)}\right), \tag{2}$$

where cd is one of the 12 Jacobi elliptic functions (see `JacobiCD` in [9]), cd^{-1} is the inverse cd Jacobi elliptic function (see `InverseJacobiCD` in [9]), K is the complete elliptic integral of the first kind (see `EllipticK` in [9]), n is the order (a positive integer), ξ is the *selectivity factor* ($\xi > 1$), and $L_n(\xi)$ is the discrimination factor defined in the Introduction. It can also be defined as the value of the elliptic rational function for $x = \xi$, that is, $L_n(\xi) = R_n(\xi, \xi)$ [4].

$R_n(\xi, x)$ can be represented by using the parametric equations

$$\begin{aligned} R_n(\xi, x) &= \operatorname{cd}\left(n\, w\, K\left(\frac{1}{L_n(\xi)}\right), \frac{1}{L_n(\xi)}\right) \\ x &= \operatorname{cd}\left(w\, K\left(\frac{1}{\xi}\right), \frac{1}{\xi}\right) \end{aligned} \tag{3}$$

where w is an intermediate variable [4].

Traditionally, $R_n(\xi, x)$ for known n, ξ, and x, can be computed as follows.

1. Find $K(1/\xi)$ using `EllipticK`.
2. Find w from the inverse cd Jacobi elliptic function.
3. Determine $K(1/L_n(\xi))$ and $L_n(\xi)$ from the *degree equation*

$$n \frac{K\left(\frac{1}{L_n(\xi)}\right)}{K\left(\sqrt{1 - \frac{1}{L_n^2(\xi)}}\right)} = \frac{K\left(\frac{1}{\xi}\right)}{K\left(\sqrt{1 - \frac{1}{\xi^2}}\right)}. \tag{4}$$

4. Find $R_n(\xi, x)$ using the cd Jacobi elliptic function.

The Chebyshev polynomial $T_n(x)$ can be derived from $R_n(\xi, x)$ for $\xi \to \infty$ because $\operatorname{cd}(\omega, 0) = \cos(\omega)$, that is,

$$\lim_{\xi \to \infty} (R_n(\xi, x)) = \cos(n \cos^{-1}(x)) = T_n(x). \tag{5}$$

The elliptic rational function in terms of the Jacobi elliptic functions can be expanded as a rational function in terms of x [4]. The explicit formulas of the first three functions are

$$R_1(\xi, x) = x \tag{6}$$

$$R_2(\xi, x) = \frac{\left(1 + \sqrt{1 - \frac{1}{\xi^2}}\right) x^2 - 1}{\left(-1 + \sqrt{1 - \frac{1}{\xi^2}}\right) x^2 + 1} \tag{7}$$

$$R_3(\xi,\, x) = x\, \frac{\left(1+\mathrm{dn}\left(\frac{2}{3}\,K\left(\frac{1}{\xi}\right),\, \frac{1}{\xi}\right)\right)^2 x^2 - \left(1+2\,\mathrm{dn}\left(\frac{2}{3}\,K\left(\frac{1}{\xi}\right),\, \frac{1}{\xi}\right)\right)}{\left(-1+\mathrm{dn}^2\left(\frac{2}{3}\,K\left(\frac{1}{\xi}\right),\, \frac{1}{\xi}\right)\right)x^2 + 1}. \tag{8}$$

In *Mathematica*, $R_3(\xi,\, x)$ can be represented as follows.

In[4]:=
```
(x ((1 + JacobiDN[2/3 EllipticK[1/ξ^2], 1/ξ^2])^2 x^2 -
      (1 + 2 JacobiDN[2/3 EllipticK[1/ξ^2], 1/ξ^2]))) /
   (1 + (-1 + JacobiDN[2/3 EllipticK[1/ξ^2], 1/ξ^2]^2) x^2) //
  TraditionalForm
```

Out[4]//TraditionalForm=

$$\frac{x\left(x^2\left(\mathrm{dn}\left(\frac{2}{3}\,K\left(\frac{1}{\xi^2}\right) \,\middle|\, \frac{1}{\xi^2}\right)+1\right)^2 - 2\,\mathrm{dn}\left(\frac{2}{3}\,K\left(\frac{1}{\xi^2}\right) \,\middle|\, \frac{1}{\xi^2}\right)-1\right)}{\left(\mathrm{dn}\left(\frac{2}{3}\,K\left(\frac{1}{\xi^2}\right) \,\middle|\, \frac{1}{\xi^2}\right)^2-1\right)x^2+1}$$

Note that `JacobiDN` requires $\frac{1}{\xi^2}$ instead of $\frac{1}{\xi}$.

The meaning of the Jacobi elliptic function notations follows: $\mathrm{cd}(u,\, k)$ means k is the modulus, $\mathrm{cd}(u \mid m)$ means $m = k^2$ is the parameter. The most common notation uses the form with k, but the elliptic functions are implemented in *Mathematica* using the form with m instead.

■ Properties

We implemented $R_n(\xi,\, x)$ in *Mathematica* as `EllipticRationalFunction[n,ξ,x]`.

The main properties of $R_n(\xi,\, x)$ follow.

- The squared function is even, $R_n^2(\xi,\, -x) = R_n^2(\xi,\, x)$.
- Over the unit interval $-1 \leq x \leq 1$, $R_n(\xi,\, x)$ is equiripple and $R_n^2(\xi,\, x) \leq 1$.
- For $x = 1$, $R_n(\xi,\, 1) = 1$.
- $R_n(\xi,\, x)$ is monotonic increasing: $1 < R_n(\xi,\, x) < R_n(\xi,\, \xi)$, in the interval $1 < x < \xi$.
- For $x \geq \xi > 1$, $|R_n(\xi,\, x)| \geq L_n(\xi) = R_n(\xi,\, \xi)$.
- The key feature of $R_n(\xi,\, x)$ is

$$R_n\,(\xi,\, x) = \frac{R_n\,(\xi,\, \xi)}{R_n\,(\xi,\, \frac{\xi}{x})}. \tag{9}$$

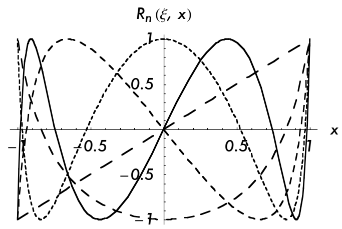
- The poles of $R_n(\xi, x)$ can be simply expressed in terms of the zeros of $R_n(\xi, x)$:

$$x_{\text{pole}} = \frac{\xi}{x_{\text{zero}}}. \tag{10}$$

The next figure illustrates the equiripple property of $R_n(\xi, x)$.

In[5]:=
```
With[{ξ = 1.1, R = EllipticRationalFunction[n, ξ, x]},
 Plot[Evaluate[Table[EllipticRationalFunction[n, ξ, x], {n, 1, 5}]],
  {x, -1, 1}, AxesLabel → {x, R},
  PlotStyle → {Dashing[{0.04}], Dashing[{0.03}],
    Dashing[{0.02}], Dashing[{0.01}], Dashing[{}]}]]
```

From In[5]:=



■ Nesting Property

Higher-order elliptic rational functions can be generated from lower-order functions by using the nesting property [4]

$$R_{mp}(\xi, x) = R_m(R_p(\xi, \xi), R_p(\xi, x)). \tag{11}$$

For example, $R_4(\xi, x) = R_2(R_2(\xi, \xi), R_2(\xi, x))$.

In[6]:=
```
With[{ξ = 2},
  R4 = EllipticRationalFunction[2, EllipticRationalFunction[2, ξ, ξ],
    EllipticRationalFunction[2, ξ, x]]];
R4 // FullSimplify // TraditionalForm
N[R4 // Together] // TraditionalForm
```

Out[7]//TraditionalForm=

$$\frac{\frac{\left(1+2\sqrt{-24+14\sqrt{3}}\right)\left(\left(2+\sqrt{3}\right)x^2-2\right)^2}{\left(\left(-2+\sqrt{3}\right)x^2+2\right)^2}-1}{\frac{\left(-1+2\sqrt{-24+14\sqrt{3}}\right)\left(\left(2+\sqrt{3}\right)x^2-2\right)^2}{\left(\left(-2+\sqrt{3}\right)x^2+2\right)^2}+1}$$

Out[8]//TraditionalForm=

$$\frac{13.8743\,x^4 - 14.373\,x^2 + 1.99484}{0.017926\,x^4 - 0.516636\,x^2 + 1.99484}$$

The corresponding nesting formula can be derived for the zeros and poles of $R_n(\xi, x)$ as shown in [4]. For $n = 2^i\,3^j$ the zeros and poles of $R_n(\xi, x)$ can be expressed symbolically in terms of ξ without using the Jacobi elliptic functions. Here is an example.

```
In[9]:=  R2 = EllipticRationalFunction[2, ξ, x];
         zeros2 = FullSimplify[Solve[Numerator[R2] == 0, x], ξ > 1 && ξ ∈ Reals]
```

$$\text{Out[10]=}\quad \left\{\left\{x \to -\frac{1}{\sqrt{1+\sqrt{1-\frac{1}{\xi^2}}}}\right\}, \left\{x \to \frac{1}{\sqrt{1+\sqrt{1-\frac{1}{\xi^2}}}}\right\}\right\}$$

```
In[11]:= poles2 = FullSimplify[Solve[Denominator[R2] == 0, x], ξ > 1 && ξ ∈ Reals]
```

$$\text{Out[11]=}\quad \left\{\left\{x \to -\frac{1}{\sqrt{1-\sqrt{1-\frac{1}{\xi^2}}}}\right\}, \left\{x \to \frac{1}{\sqrt{1-\sqrt{1-\frac{1}{\xi^2}}}}\right\}\right\}$$

For orders $n \neq 2^i\, 3^j$, $R_n(\xi, x)$ cannot be expressed symbolically without the Jacobi elliptic functions or the like.

■ Algorithm

In this section we implement the elliptic rational functions in *Mathematica*.

□ Elliptic Rational Functions

First, we define the first-order function $R_1(\xi, x) = x$.

```
In[12]:= EllipticRationalFunction[1, ξ_, x_] := x;
```

We use the nesting property and the closed-form expressions [4] for orders $n = 2^i\, 3^j$.

```
In[13]:= EllipticRationalFunction[n_?EvenQ, ξ_, x_] :=
```

$$\text{Module}\left[\{t\},\ t = \sqrt{1 - \frac{1}{\text{DiscriminationFactor}\left[\frac{n}{2}, \xi\right]^2}};\ \frac{(t+1)\,\text{EllipticRationalFunction}\left[\frac{n}{2}, \xi, x\right]^2 - 1}{(t-1)\,\text{EllipticRationalFunction}\left[\frac{n}{2}, \xi, x\right]^2 + 1}\right];$$

```
In[14]:= EllipticRationalFunction[n_?(Mod[#1, 3] == 0 &), ξ_, x_] :=
```

$$\text{Module}\Bigg[\{b, c, r, y\},\ y = 1 - \frac{2}{\text{DiscriminationFactor}\left[\frac{n}{3}, \xi\right]^2};\ b = (1-y^2)^{1/3};\ c = \sqrt{1+b+b^2};\ r = \frac{1}{2}\left(\frac{y}{c} + \sqrt{2+b+2c} - 1\right);$$
$$\left(\text{EllipticRationalFunction}\left[\frac{n}{3}, \xi, x\right]\left((r+1)^2\,\text{EllipticRationalFunction}\left[\frac{n}{3}, \xi, x\right]^2 - (1+2r)\right)\right) \Big/ \left((r^2-1)\,\text{EllipticRationalFunction}\left[\frac{n}{3}, \xi, x\right]^2 + 1\right)\Bigg];$$

We use the Jacobi elliptic function JacobiSN and the complete elliptic integral of the first kind EllipticK for orders $n \neq 2^i\, 3^j$.

In[15]:=
```
EllipticRationalFunction[n_Integer, ξ_, x_] :=
  EllipticRationalFunctionJacobi[n, ξ, x];
```

In[16]:=
```
EllipticRationalFunctionJacobi[n_Integer, ξ_, x_] :=
  Module[{p, R, t, z}, z = EllipticRationalFunctionZeroJacobi[n, 1/ξ];
   p = EllipticRationalFunctionPoleJacobi[n, 1/ξ];
   R = (Times @@ (t - z))/(Times @@ (t - p)); (R /. t → x)/(R /. t → 1)];
```

□ Discrimination Factor

The first-order discrimination factor equals the selectivity factor.

In[17]:=
```
DiscriminationFactor[1, ξ_] := ξ;
```

We use the nesting property and the closed-form expressions [4] for orders $n = 2^i\, 3^j$.

In[18]:=
```
DiscriminationFactor[n_?EvenQ, ξ_] :=
  Module[{b}, b = DiscriminationFactor[n/2, ξ]; (b + Sqrt[b^2 - 1])^2];
```

In[19]:=
```
DiscriminationFactor[n_?(Mod[#1, 3] == 0 &), ξ_] := Module[
   {b, c, r, y}, y = 1 - 2/DiscriminationFactor[n/3, ξ]^2; b = (1 - y^2)^(1/3);
   c = Sqrt[1 + b + b^2]; r = 1/2 (y/c + Sqrt[2 + b + 2 c] - 1); Sqrt[(1 + 2 r)^3/((1 - r)^3 (1 + r))]];
```

We use the Jacobi elliptic function JacobiSN and the complete elliptic integral of the first kind EllipticK for orders $n \neq 2^i\, 3^j$.

In[20]:=
```
DiscriminationFactor[n_Integer, ξ_] :=
  DiscriminationFactorJacobi[n, ξ];
```

In[21]:=
```
DiscriminationFactorJacobi[n_Integer, ξ_] :=
  EllipticRationalFunctionJacobi[n, ξ, ξ];
```

□ Zeros and Poles

The zeros and poles of $R_n(\xi, x)$ for orders $n \neq 2^i\, 3^j$ are computed in terms of the Jacobi elliptic function JacobiSN and the complete elliptic integral of the first kind EllipticK.

```
In[22]:= EllipticRationalFunctionZeroJacobi[n_Integer, k_] := Module[{t, z},
           t = (JacobiSN[EllipticK[k^2] + EllipticK[k^2] (2 #1 - 1)/n, k^2] &) /@
             Range[Floor[n/2]]; If[EvenQ[n],
            z = Join[-t, Reverse[t]], z = Join[-t, {0}, Reverse[t]]]; z];

In[23]:= EllipticRationalFunctionPoleJacobi[n_Integer, k_] := Module[{t},
           t = (1/(k JacobiSN[EllipticK[k^2] + EllipticK[k^2] (2 #1 - 1)/n,
                  k^2]) &) /@ Range[Floor[n/2]]; Join[-Reverse[t], t]];
```

Closed-form expressions for the zeros and poles of $R_n(\xi, x)$ exist [4] for orders $n = 2^i\, 3^j$ and these formulas do not require Jacobi functions.

□ Traditional Forms

```
In[24]:= EllipticRationalFunction /:
          MakeBoxes[EllipticRationalFunction[n_, a_, x_], TraditionalForm] :=
          RowBox[{SubscriptBox["R", MakeBoxes[n, TraditionalForm]],
            "(", MakeBoxes[a, TraditionalForm], ",",
            MakeBoxes[x, TraditionalForm], ")"}]

In[25]:= DiscriminationFactor /:
          MakeBoxes[DiscriminationFactor[n_, a_], TraditionalForm] :=
          RowBox[{SubscriptBox["L", MakeBoxes[n, TraditionalForm]],
            "(", MakeBoxes[a, TraditionalForm], ")"}]
```

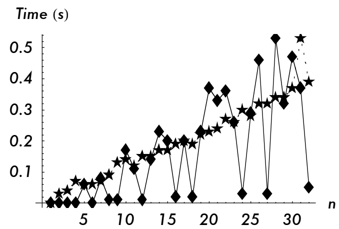
■ Timing

We time how long it takes to calculate $R_n(\xi, x)$ using our symbolic algorithm, `EllipticRationalFunction[n,ξ,x]`, and the numeric algorithm [1] that we implemented in *Mathematica* as `EllipticRationalFunctionJacobi[n,ξ,x]`. The following figure shows that the proposed algorithm (solid line) runs much faster than the traditional algorithm for orders $n = 2^i\, 3^j$.

```
In[26]:= With[{ξ = N[11/10, 64]}, t1List = {}; t2List = {};
          Do[t1 = Timing[N[EllipticRationalFunction[n, ξ, x]]];
           t2 = Timing[N[EllipticRationalFunctionJacobi[n, ξ, x]]];
           AppendTo[t1List, t1[[1]]]; AppendTo[t2List, t2[[1]]];, {n, 1, 32}]]
```

In[27]:=
```
<< "Graphics`MultipleListPlot`"
MultipleListPlot[{t1List/Second, t2List/Second},
 PlotJoined → True, AxesLabel → {n, "Time (s)"}]
```

From In[27]:=



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About the Authors

Miroslav D. Lutovac is a chief scientist at the Institute for Telecommunications and Electronics and a professor at the University of Belgrade, Serbia. His research interests include the theory and implementation of analog and digital signal processing, and the symbolic analysis and synthesis of multiplierless and multirate digital systems.

Dejan V. Tosic is a professor in the School of Electrical Engineering at the University of Belgrade, Serbia. He has focused his research on creating a framework for the symbolic analysis of circuits and systems that is suitable for research as well as industrial and educational applications. He is developing automation tools for optimizing the design and synthesis of analog and digital systems.

Lutovac and Tosic are coauthors (with Brian Evans) of the book *Filter Design for Signal Processing Using MATLAB and Mathematica* [4]. They have developed *SchematicSolver*, a *Mathematica* application for mouse-driven interactive drawing of systems and for solving and implementing said systems.

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