Tricks of the Trade

Edited by Paul Abbott

This is a column of programming tricks and techniques, most of which, we hope, will be contributed by our readers, either directly as submissions to *The Mathematica Journal* or as an edited answer to a question posted in the *Mathematica* news group, comp.soft-sys.math.mathematica.

■ Nondecreasing Function

Consider determining real coefficients a and b such that $f_{a,b}(x) = x^4 + 2\,a\,x^3 + b\,x^2 + (1 - 3\,a)\,x + 4$ is *nondecreasing* for $-1 \le x \le 1$.

In Version 5.1, applying **Resolve** to the requirement $f'_{a,b}(x) \ge 0$ yields the answer.

In[1]:= $\mathbf{f_{a\_,b\_}(x\_) = x^4 + 2\,a\,x^3 + b\,x^2 + (1 - 3\,a)\,x + 4;}$

In[2]:= **coefs = Resolve[**$\forall_{x,-1\le x\le 1}\ f'_{a,b}(x) \ge 0$**]**

Out[2]= $a = 0 \bigwedge b = -\frac{3}{2}$

The derivative is clearly nonnegative for $-1 \le x \le 1$.

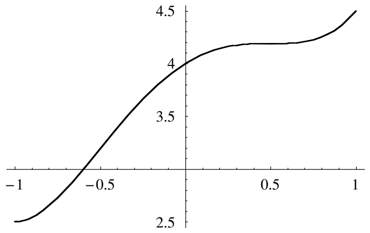
In[3]:= $f'_{a,b}(x)$ **/. ToRules[coefs] // Factor**

Out[3]= $(x + 1)\,(2\,x - 1)^2$

A plot gives visual confirmation that the function is nondecreasing.

In[4]:= **Plot[Evaluate[**$f_{a,b}(x)$ **/. ToRules[coefs]], {**x**, −1, 1}]**

From In[4]:=



■ Proofs of Inequalities

Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$s_n = \frac{x_1}{x_1^2+1} + \frac{x_2}{x_1^2+x_2^2+1} + \cdots + \frac{x_n}{x_1^2+x_2^2+\cdots+x_n^2+1} < \sqrt{n}$$

for all integer $n \geq 1$.

Define s_n.

In[1]:= $$s_{n\_} := \sum_{i=1}^{n} \frac{x_i}{\sum_{j=1}^{i} x_j^2 + 1}$$

First, we use **NMaximize** to examine the inequality for $1 \leq n \leq 20$.

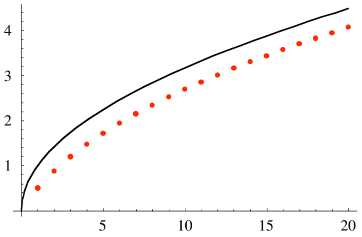
In[2]:= **pts = Table[First[NMaximize[s_n, Table[{x_i, −5, 5}, {i, n}]]], {n, 20}]**

Out[2]=
```
{0.5, 0.880086, 1.19591, 1.47044, 1.71578, 1.93915,
  2.14528, 2.33746, 2.51805, 2.68883, 2.85119, 3.0062, 3.15475,
  3.29756, 3.43522, 3.56823, 3.69702, 3.82195, 3.94334, 4.06148}
```

In[3]:= **<< Graphics`;**

In[4]:= **DisplayTogether[Plot[$\sqrt{n}$, {n, 0, 20}],**
ListPlot[pts, PlotStyle → {Hue[1], AbsolutePointSize[2]}]]

From In[4]:=



From the form and statement of the inequality, a proof by induction [mathworld.wolfram.com/PrincipleofMathematicalInduction.html] looks like the way to proceed. Leon Aigret (aigret@myrealbox.com) presented the following inductive proof on the alt.math.recreational newsgroup, here restated using *Mathematica*.

For $n = 1$, the proof is trivial.

In[5]:= **Reduce[s_1 < 1]**

Out[5]= $x_1 \in \mathbb{R}$

We need to show that $s_{n-1} < \sqrt{n-1} \Rightarrow s_n < \sqrt{n}$ for all $n \geq 2$. Consider the transformation $x_k \to \sqrt{1 + x_1^2}\; y_{k-1}$ for $k \geq 2$. Here is the result with $n = 4$.

In[6]:= $\mathbf{Factor\ /@}\left(s_4\ /.\ x_{\mathrm{k\_/;}k\geq 2} \to \sqrt{1+x_1^2}\ y_{k-1}\right)$

Out[6]= $\dfrac{x_1}{x_1^2+1}+\dfrac{y_1}{\sqrt{x_1^2+1}\ (y_1^2+1)}+\dfrac{y_2}{\sqrt{x_1^2+1}\ (y_1^2+y_2^2+1)}+\dfrac{y_3}{\sqrt{x_1^2+1}\ (y_1^2+y_2^2+y_3^2+1)}$

In this result we recognize s_3 involving y instead of x.

In[7]:= $\% == s_1+\dfrac{1}{\sqrt{x_1^2+1}}\ (s_3\ /.\ x \to y)\ //\ \mathbf{Simplify}$

Out[7]= True

In general we have

$$s_n(x) \longrightarrow \frac{x_1}{x_1^2+1}+\frac{y_1}{\sqrt{x_1^2+1}\ (y_1^2+1)}+\frac{y_2}{\sqrt{x_1^2+1}\ (y_1^2+y_2^2+1)}+\cdots+$$
$$\frac{y_{n-1}}{\sqrt{x_1^2+1}\ (y_1^2+y_2^2+\cdots+y_{n-1}^2)+1}=\frac{x_1}{x_1^2+1}+\frac{1}{\sqrt{x_1^2+1}}\ s_{n-1}(y).$$

So, assuming that $s_{n-1}<\sqrt{n-1}$, we need to prove that

$$\frac{x_1}{x_1^2+1}+\frac{\sqrt{n-1}}{\sqrt{x_1^2+1}}<\sqrt{n}\,.$$

It is straightforward to show that this inequality holds for $n \geq 1$ and $x_1 \in \mathbb{R}$ so we have completed the proof.

In[8]:= $\mathbf{Reduce}\left[\dfrac{x_1}{x_1^2+1}+\dfrac{\sqrt{n-1}}{\sqrt{x_1^2+1}}<\sqrt{n},\ \mathbb{R}\right]$

Out[8]= $n \geq 1$

Andrzej Kozlowski (www.mimuw.edu.pl/~akoz) proved the following stronger statement:

$$t_n=\sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{i} x_j^2+1}\right)^2<1$$

for every positive integer n.

It is easy to see that this statement implies the original inequality using Cauchy's inequality [mathworld.wolfram.com/CauchysInequality.html],

$$\left(\sum_{i=1}^{n} a_i\, b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Setting $b_i = 1$,

$$\left(\sum_{i=1}^{n} a_i^2\right) \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2,$$

and hence

$$1 > \sum_{i=1}^{n}\left(\frac{x_i}{\sum_{j=1}^{i} x_j^2 + 1}\right)^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n}\frac{x_i}{\sum_{j=1}^{i} x_j^2 + 1}\right)^2 \Rightarrow$$

$$\sum_{i=1}^{n}\frac{x_i}{\sum_{j=1}^{i} x_j^2 + 1} < \sqrt{n}\,.$$

Moreover, the proof is easier since the inductive step is now trivial.

$$t_n(x) \longrightarrow \frac{x_1^2}{(x_1^2+1)^2} +$$

$$\frac{1}{x_1^2+1}\left(\frac{y_1^2}{(y_1^2+1)^2} + \frac{y_2^2}{(y_1^2+y_2^2+1)^2} + \cdots + \frac{y_{n-1}^2}{(y_1^2+y_2^2+\cdots+y_{n-1}^2+1)^2}\right) =$$

$$\frac{x_1^2}{(x_1^2+1)^2} + \frac{t_{n-1}(y)}{x_1^2+1}.$$

So, assuming that $t_{n-1} < 1$, we need to prove that

$$t_n < \frac{x_1^2}{(x_1^2+1)^2} + \frac{1}{x_1^2+1} < 1.$$

This is clearly true for $x_1 \neq 0$.

In[9]:= $\mathbf{Reduce}\left[\frac{x_1^2}{(x_1^2+1)^2} + \frac{1}{x_1^2+1} < 1, \mathbb{R}\right]$

Out[9]= $x_1 < 0 \lor x_1 > 0$

If $x_1 = 0$, then $t_n(x) = t_{n-1}(y) < 1$ so the proof is complete.

In addition, the inequality leads to some intriguing observations since it implies that the sums, considered as functions on the real line, are bounded and attain their maxima. So it is natural to consider the functions $f_n(x)$ obtained by setting all the $x_i = x_j$.

In[10]:= $f_{\text{n\_}}(\text{x\_}) = \sum_{i=1}^{n}\left(\frac{x}{i\,x^2+1}\right)^2$

Out[10]= $\frac{\psi^{(1)}(1+\frac{1}{x^2}) - \psi^{(1)}(n+\frac{1}{x^2}+1)}{x^2}$

The limit as $n \to \infty$ can be computed in closed form.

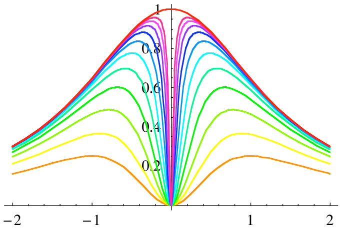
In[11]:= $f_\infty(\mathrm{x\_}) = \sum_{i=1}^{\infty} \left(\frac{x}{i\,x^2+1}\right)^2$

Out[11]= $\frac{\psi^{(1)}\left(\frac{x^2+1}{x^2}\right)}{x^2}$

It is interesting to visualize the convergence of $f_n \to f_\infty$.

In[12]:= **Plot[Evaluate[Join[Table[f_{2^n}(x), {n, 0, 10}], {f_∞(x)}]], {x, −2, 2}, PlotStyle → Table[Hue[i / 12], {i, 12}]]**

From In[12]:=



■ Explicit Roots of Transcendental Equations

Cauchy's integral theorem [mathworld.wolfram.com/CauchyIntegralTheorem.html] states that if $f(z)$ is analytic in some simply connected region $\mathcal{R}$, then

$$\oint_\gamma f(z)\,dz = 0 \tag{1}$$

for any closed contour γ completely contained in $\mathcal{R}$. The roots of a function $g(z)$ can be determined by locating the singularities of the reciprocal of the function, $h(z) = 1/g(z)$. If $h(z)$ has a simple pole at z_0, then $f(z) = (z - z_0)\,h(z)$ is analytic. Luck and Stevens [1] use this to obtain an *explicit* expression for the root z_0 of $g(z)$,

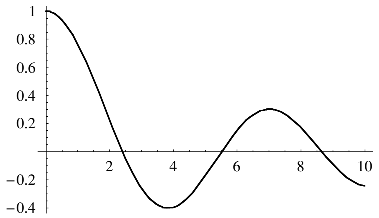
$$\oint_\gamma (z - z_0)\,h(z)\,dz = 0 \Rightarrow z_0 = \frac{\oint_\gamma z\,h(z)\,dz}{\oint_\gamma h(z)\,dz}, \tag{2}$$

where the contour γ contains the single pole at z_0.

For example, consider computing the roots of $J_0(z)$. The roots lie on the real axis.

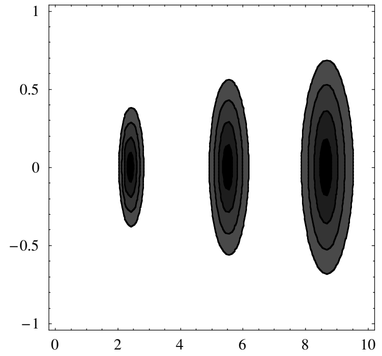
In[1]:= **Plot[$\mathcal{J}_0(z)$, {z, 0, 10}]**

From In[1]:=



In[2]:= **ContourPlot[|$\mathcal{J}_0(x + i\, y)$|, {x, 0, 10}, {y, −1, 1},
Contours → 0.05 Range[4], PlotPoints → 100]**

From In[2]:=



To determine the first root, we use equation (2), integrating around an arbitrary contour enclosing just this root.

In[3]:= **h(z_) = 1 / $\mathcal{J}_0(z)$;**

In[4]:= $$j_{0,1} = \frac{\textbf{NIntegrate}[z\,h(z),\ \{z,\ 1,\ 2-i,\ 3,\ 2+i,\ 1\}]}{\textbf{NIntegrate}[h(z),\ \{z,\ 1,\ 2-i,\ 3,\ 2+i,\ 1\}]}\ \textbf{// Chop}$$

Out[4]= 2.40483

We verify that this value is correct to machine precision.

In[5]:= $\mathcal{J}_0(j_{0,1})$

Out[5]= -4.89674×10^{-16}

Alternatively, evaluating both integrals around the same circular contour

$$z = a + r\, e^{i\theta} \Rightarrow dz = i\, r\, e^{i\theta}\, d\theta,$$

equation (2) becomes

$$z_0 = a + r\left(\frac{\int_0^{2\pi} f(\theta)\, e^{2 i\theta}\, d\theta}{\int_0^{2\pi} f(\theta)\, e^{i\theta}\, d\theta}\right), \tag{3}$$

where $f(\theta) = h(a + r\,e^{i\,\theta})$. The values of a and r do not matter as long as the contour circumscribes the root.

To determine the second zero of $\mathcal{J}_0(z)$, we choose a circle centered at $a = 5$ with radius $r = 1$ that circumscribes this root and no other.

In[6]:= $\mathbf{a = 5;\ r = 1;\ f(\theta\_) := h(a + r\,e^{i\,\theta});}$

We use equation (3) to determine $j_{0,2}$.

In[7]:= $$j_{0,2} = a + r\,\frac{\textbf{NIntegrate}[f(\theta)\,e^{2\,i\,\theta},\ \{\theta,\ 0,\ 2\,\pi\}]}{\textbf{NIntegrate}[f(\theta)\,e^{i\,\theta},\ \{\theta,\ 0,\ 2\,\pi\}]}\ \text{// }\textbf{Chop}$$

Out[7]= 5.52008

This is an excellent approximation to the second root.

In[8]:= $\mathcal{J}_0(j_{0,2})$

Out[8]= 8.36295×10^{-16}

The nth complex Fourier coefficient is defined by [mathworld.wolfram.com/FourierSeries.html]

$$c_n = \frac{1}{2\,\pi}\int_0^{2\,\pi} f(\theta)\,e^{i\,n\,\theta}\,d\theta,$$

so equation (3) reduces to

$$z_0 = a + r\,\frac{c_2}{c_1}, \tag{4}$$

involving a simple ratio of Fourier coefficients.

To determine the third zero of $\mathcal{J}_0(z)$, we choose a circle centered at $a = 9$ with radius $r = 1$ that circumscribes this root and no other, computing the Fourier coefficients *approximately* using **Fourier**, sampling $f(\theta)$ uniformly over $[0, 2\,\pi]$ just 16 times.

In[9]:= $\mathbf{a = 9;\ cs = Rest\Big[Fourier\Big[Table\Big[f(\theta),\ \Big\{\theta,\ 0,\ 2\,\pi - \frac{\pi}{8},\ \frac{\pi}{8}\Big\}\Big]\Big]\Big]}$

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Out[9]= {-14.7356, 5.10251, -1.76687, 0.611787, -0.211934,
          0.0731576, -0.0260457, 0.00728334, -0.00824435, -0.0100118,
          -0.042844, -0.0765791, -0.346537, -0.463324, -2.64574}
```

We use equation (4) to determine $j_{0,3}$.

In[10]:= $$j_{0,3} = a + r\,\frac{\textbf{cs}[\![2]\!]}{\textbf{cs}[\![1]\!]}$$

Out[10]= 8.65373

Then we check that this value is a good approximation to the third root.

In[11]:= $\mathcal{J}_0(j_{0,3})$

Out[11]= -7.16756×10^{-8}

Increasing the number of sample points improves the accuracy of roots computed via **Fourier**.

■ References

[1] R. Luck and J. W. Stevens, "Explicit Solutions for Transcendental Equations," *SIAM Review*, **44**(2), 2002 pp. 227–233.

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