

Neighbourhoods of Independence for Random Processes via Information Geometry

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In this article we consider the Freund family of distributions as a 4-manifold equipped with Fisher information as Riemannian metric and derive the induced α -geometry, that is, the α -Ricci curvature, the α -scalar curvature, and so forth. We show that the Freund manifold has a positive constant 0-scalar curvature, so geometrically it constitutes part of a sphere. We examine special cases as submanifolds and discuss their geometrical structures; via one submanifold we provide examples of neighbourhoods of the independent case for bivariate distributions having identical exponential marginals. Thus, since exponential distributions complement Poisson point processes, we obtain a means to discuss the neighbourhood of independence for random processes. Our approach using *Mathematica* handles the α -geometry calculations and graphics effectively and could be transferred to other distribution families.

■ 1. Differential Geometry of the Freund 4-Manifold F

In [1] we proved that every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions in the subspace topology of \mathbb{R}^3 , using information geometry (see Amari et al. [2, 3]) and the affine immersion of Dodson and Matsuzoe [4]. As part of a study of the information geometry of gamma and bivariate gamma stochastic processes [5, 6, 7], we have calculated the geometry of the family of Freund bivariate mixture exponential density functions. The importance of this family lies in the fact that exponential distributions represent intervals between events for Poisson processes on the real line, and the Freund family provides models for bivariate processes with positive and negative covariance.

The Freund family of distributions becomes a Riemannian 4-manifold with the Fisher information metric, and we derive the induced α -geometry, that is, the α -Ricci curvature, the α -scalar curvature, and so forth. We also show that the Freund manifold has a positive constant 0-scalar curvature, so geometrically it constitutes part of a sphere.

□ 1.1. Freund Bivariate Mixture Exponential Distributions

This model was one of the first bivariate distributions to be obtained from reliability considerations. Freund [8] introduced a bivariate exponential mixture distribution arising from the following situation. Suppose that an instrument has two components, A and B , with lifetimes X and Y having density functions (when both components are in operation) $f_X(x) = \alpha_1 e^{-\alpha_1 x}$; $f_Y(y) = \alpha_3 e^{-\alpha_3 y}$, ($\alpha_1, \alpha_3 > 0$; $x, y > 0$). Then X and Y are dependent, in that a failure of either component changes the parameter of the life distribution of the other component. Thus, when A fails, the parameter for Y becomes α_4 ; when B fails, the parameter for X becomes α_2 . There is no other dependence. The joint density function of X and Y is

$$f(x, y) = \begin{cases} \alpha_1 \alpha_4 e^{-\alpha_4 y - (\alpha_1 + \alpha_3 - \alpha_4)x} & \text{for } 0 < x < y \\ \alpha_3 \alpha_2 e^{-\alpha_2 x - (\alpha_1 + \alpha_3 - \alpha_2)y} & \text{for } 0 < y < x, \end{cases} \quad (1)$$

where $\alpha_i > 0$ ($i = 1, 2, 3, 4$).

Define the functions f_1 and f_2 :

$$f_1 := \alpha_1 \alpha_4 e^{-\alpha_4 y - (\alpha_1 + \alpha_3 - \alpha_4)x};$$

and

$$f_2 := \alpha_2 \alpha_3 e^{-\alpha_2 x - (\alpha_1 + \alpha_3 - \alpha_2)y};$$

Provided that $\alpha_1 + \alpha_3 \neq \alpha_2$, the marginal density function of X is, according to Freund,

$$f_X(x) = \left(\frac{\alpha_3}{\alpha_1 + \alpha_3 - \alpha_2} \right) \alpha_2 e^{-\alpha_2 x} + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_3 - \alpha_2} \right) (\alpha_1 + \alpha_3) e^{-(\alpha_1 + \alpha_3)x}, \quad (2)$$

$$x \geq 0,$$

and provided that $\alpha_1 + \alpha_3 \neq \alpha_4$, the marginal density function of Y is

$$f_Y(y) = \left(\frac{\alpha_1}{\alpha_1 + \alpha_3 - \alpha_4} \right) \alpha_4 e^{-\alpha_4 y} + \left(\frac{\alpha_3 - \alpha_4}{\alpha_1 + \alpha_3 - \alpha_4} \right) (\alpha_1 + \alpha_3) e^{-(\alpha_1 + \alpha_3)y}, \quad (3)$$

$$y \geq 0.$$

We can see that the marginal density functions are, in general, not exponential but rather mixtures of exponential distributions if $\alpha_i > \alpha_{i+1}$ ($i = 1, 3$); otherwise, they are weighted averages. For this reason, this system of distributions should be termed bivariate mixture exponential distributions rather than simply bivariate exponential distributions. The marginal functions $f_X(x)$ and $f_Y(y)$ are exponential distributions only in the special case $\alpha_i = \alpha_{i+1}$ ($i = 1, 3$).

Freund discussed the statistics of the special case when $\alpha_1 + \alpha_3 = \alpha_2 = \alpha_4$ and obtained the following joint density function

$$f(x, y) = \begin{cases} \alpha_1 (\alpha_1 + \alpha_3) e^{-(\alpha_1 + \alpha_3)y} & \text{for } 0 < x < y \\ \alpha_3 (\alpha_1 + \alpha_3) e^{-(\alpha_1 + \alpha_3)x} & \text{for } 0 < y < x \end{cases} \quad (4)$$

with marginal density functions:

$$f_X(x) = (\alpha_1 + \alpha_3 (\alpha_1 + \alpha_3) x) e^{-(\alpha_1 + \alpha_3)x}, \quad x \geq 0 \quad (5)$$

$$f_Y(y) = (\alpha_3 + \alpha_1 (\alpha_1 + \alpha_3) y) e^{-(\alpha_1 + \alpha_3)y}, \quad y \geq 0. \quad (6)$$

The covariance and correlation coefficient of X and Y were derived by Freund, as follows:

$$\text{Cov}(X, Y) = \frac{\alpha_2 \alpha_4 - \alpha_1 \alpha_3}{\alpha_2 \alpha_4 (\alpha_1 + \alpha_3)^2}, \quad (7)$$

$$\rho(X, Y) = \frac{\alpha_2 \alpha_4 - \alpha_1 \alpha_3}{\sqrt{(\alpha_1^2 + 2 \alpha_1 \alpha_3 + \alpha_4^2)(\alpha_3^2 + 2 \alpha_1 \alpha_3 + \alpha_2^2)}}. \quad (8)$$

Note that in general $\frac{-1}{3} < \rho(X, Y) < 1$. The correlation coefficient $\rho(X, Y) \rightarrow 1$ when $\alpha_2, \alpha_4 \rightarrow \infty$, and $\rho(X, Y) \rightarrow \frac{-1}{3}$ when $\alpha_1 = \alpha_3$ and $\alpha_2, \alpha_4 \rightarrow 0$. In many applications, $\alpha_{i+1} > \alpha_i$ ($i = 1, 3$), that is, lifetime tends to be shorter when the other component is out of action. In such cases the correlation is positive.

□ 1.2. Fisher Information Metric

Proposition 1.1 Let F be the set of Freund bivariate mixture exponential distributions, that is,

$$F = \left\{ f : f(x, y; \alpha_i) = \begin{cases} \alpha_1 \alpha_4 e^{-\alpha_4 y - (\alpha_1 + \alpha_3 - \alpha_4)x} & \text{for } 0 < x < y \\ \alpha_3 \alpha_2 e^{-\alpha_2 x - (\alpha_1 + \alpha_3 - \alpha_2)y} & \text{for } 0 < y < x \end{cases}, (i = 1, 2, 3, 4) \right\}. \quad (9)$$

Then we have:

1. Identifying $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ as a local coordinate system, F can be regarded as a 4-manifold.
2. F is a Riemannian space and the Fisher information matrix $G = [g_{ij}]$, where

$$g_{ij} = \int_0^\infty \int_0^\infty \frac{\partial^2 \log f}{\partial \alpha_i \partial \alpha_j} \cdot f \, dx \, dy, \quad (i = 1, 2, 3, 4)$$

is given by

$$\begin{aligned}
 \text{In}[3]:= & \mathbf{g} = \text{Outer}[\text{FullSimplify}[\int_0^\infty \int_x^\infty -f_1 \text{Simplify}[\frac{\partial^2 \log(f_1)}{\partial \#1 \partial \#2}]] dy dx + \\
 & \int_0^\infty \int_0^x -f_2 \text{Simplify}[\frac{\partial^2 \log(f_2)}{\partial \#1 \partial \#2}]] dy dx, \\
 & \{\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 > 0\} \&, \\
 & \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}] \\
 \text{Out}[3]= & \begin{pmatrix} \frac{1}{\alpha_1^2 + \alpha_3} \alpha_1 & 0 & 0 & 0 \\ 0 & \frac{\alpha_3}{\alpha_2^2 (\alpha_1 + \alpha_3)} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_3^2 + \alpha_1} \alpha_3 & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{(\alpha_1 + \alpha_3) \alpha_4^2} \end{pmatrix}
 \end{aligned}$$

□ **1.3. α -geometry**

In this section we provide the α -connection and various α -curvature objects of the Freund 4-manifold F : the α -curvature tensor, the α -sectional curvatures, the α -mean curvatures, the α -Ricci tensor, and the α -scalar curvature.

1.3.1. α -connection

For each $\alpha \in \mathbb{R}$, the α ($\nabla^{(\alpha)}$)-connection is the torsion-free affine connection with components:

$$\begin{aligned}
 \Gamma_{ij,k}^{(\alpha)} = & \int_0^\infty \int_0^\infty \left(\frac{\partial^2 \log(f)}{\partial \alpha_i \partial \alpha_j} \frac{\partial \log(f)}{\partial \alpha_k} + \frac{(1-\alpha)}{2} \frac{\partial \log(f)}{\partial \alpha_i} \frac{\partial \log(f)}{\partial \alpha_j} \frac{\partial \log(f)}{\partial \alpha_k} \right) \cdot f \quad (10) \\
 & dx dy, \quad (i, j, k = 1, 2, 3, 4).
 \end{aligned}$$

So we have an affine connection ($\nabla^{(\alpha)}$) defined by

$$\langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle = \Gamma_{ij,k}^{(\alpha)}, \tag{11}$$

where $g = \langle , \rangle$ is the Fisher information metric.

Note that the 0-connection is the Riemannian connection with respect to the Fisher metric.

Proposition 1.2 The functions $\Gamma_{ij,k}^{(\alpha)}$, ($i, j, k = 1, 2, 3, 4$) are given by

$$\begin{aligned}
 \text{In[4]:= } & \Gamma = \text{FullSimplify}[\text{Assuming}[\text{Re}[\alpha_1 + \alpha_3] > 0 \ \&\& \ \text{Re}[\alpha_2] > 0, \\
 & \text{Table}[\int_0^\infty \int_x^\infty f_1 \text{Simplify}[\frac{1}{2} (1 - \alpha) \frac{\partial \log(f_1)}{\partial \alpha_i} \frac{\partial \log(f_1)}{\partial \alpha_j} + \frac{\partial^2 \log(f_1)}{\partial \alpha_i \partial \alpha_j}] \\
 & \quad \frac{\partial \log(f_1)}{\partial \alpha_k} \, d y \, d x + \\
 & \quad \int_0^\infty \int_0^x f_2 \text{Simplify}[\frac{1}{2} (1 - \alpha) \frac{\partial \log(f_2)}{\partial \alpha_i} \frac{\partial \log(f_2)}{\partial \alpha_j} + \frac{\partial^2 \log(f_2)}{\partial \alpha_i \partial \alpha_j}] \\
 & \quad \frac{\partial \log(f_2)}{\partial \alpha_k} \, d y \, d x, \{i, 4\}, \{j, 4\}, \{k, 4\}]]];
 \end{aligned}$$

For example, the component $\Gamma_{11,1}^{(\alpha)}$ is given by

$$\begin{aligned}
 \text{In[5]:= } & \Gamma[[1, 1, 1]] \\
 \text{Out[5]= } & \frac{2(\alpha - 1)\alpha_1 - (\alpha + 1)\alpha_3}{2\alpha_1^2(\alpha_1 + \alpha_3)^2}
 \end{aligned}$$

Proposition 1.3 By solving the equations $\Gamma_{ij,k}^{(\alpha)} = \sum_{b=1}^4 g_{kb} \Gamma_{ij}^b$, ($k = 1, 2, 3, 4$), we obtain the components of $\nabla^{(\alpha)}$:

$$\begin{aligned}
 \text{In[6]:= } & \text{Table}[\text{FullSimplify}[\text{Solve}[\\
 & \Gamma[[i, j, 1]] == \sum_{b=1}^4 g[[1, b]] \Gamma^b(i, j) \wedge \Gamma[[i, j, 2]] == \sum_{b=1}^4 g[[2, b]] \Gamma^b(i, j) \wedge \\
 & \Gamma[[i, j, 3]] == \sum_{b=1}^4 g[[3, b]] \Gamma^b(i, j) \wedge \Gamma[[i, j, 4]] == \sum_{b=1}^4 g[[4, b]] \Gamma^b(i, j), \\
 & \{\Gamma^1(i, j), \Gamma^2(i, j), \Gamma^3(i, j), \Gamma^4(i, j)\}], \{i, 4\}, \{j, 4\}];
 \end{aligned}$$

So we obtain:

$$\begin{aligned}
 \text{In[7]:= } & \Gamma^1 = \begin{pmatrix} \frac{1}{2} \left(\frac{3\alpha-1}{\alpha_1+\alpha_3} - \frac{\alpha+1}{\alpha_1} \right) & 0 & \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & 0 \\ 0 & \frac{(\alpha+1)\alpha_1\alpha_3}{2\alpha_2^2(\alpha_1+\alpha_3)} & 0 & 0 \\ \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & 0 & \frac{(\alpha+1)\alpha_1}{2\alpha_3(\alpha_1+\alpha_3)} & 0 \\ 0 & 0 & 0 & -\frac{(\alpha+1)\alpha_1\alpha_3}{2(\alpha_1+\alpha_3)\alpha_4^2} \end{pmatrix}; \\
 \text{In[8]:= } & \Gamma^2 = \begin{pmatrix} 0 & \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & 0 & 0 \\ \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & \frac{\alpha-1}{\alpha_2} & -\frac{(\alpha-1)\alpha_1}{2\alpha_3(\alpha_1+\alpha_3)} & 0 \\ 0 & -\frac{(\alpha-1)\alpha_1}{2\alpha_3(\alpha_1+\alpha_3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};
 \end{aligned}$$

$$\begin{aligned}
 \text{In[9]:= } \Gamma^3 &= \begin{pmatrix} \frac{(\alpha+1)\alpha_3}{2\alpha_1(\alpha_1+\alpha_3)} & 0 & \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & 0 \\ 0 & -\frac{(\alpha+1)\alpha_1\alpha_3}{2\alpha_2^2(\alpha_1+\alpha_3)} & 0 & 0 \\ \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & 0 & \frac{1}{2}\left(\frac{3\alpha-1}{\alpha_1+\alpha_3} - \frac{\alpha+1}{\alpha_3}\right) & 0 \\ 0 & 0 & 0 & \frac{(\alpha+1)\alpha_1\alpha_3}{2(\alpha_1+\alpha_3)\alpha_4^2} \end{pmatrix}; \\
 \text{In[10]:= } \Gamma^4 &= \begin{pmatrix} 0 & 0 & 0 & -\frac{(\alpha-1)\alpha_3}{2\alpha_1(\alpha_1+\alpha_3)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha-1}{2(\alpha_1+\alpha_3)} \\ -\frac{(\alpha-1)\alpha_3}{2\alpha_1(\alpha_1+\alpha_3)} & 0 & \frac{\alpha-1}{2(\alpha_1+\alpha_3)} & \frac{\alpha-1}{\alpha_4} \end{pmatrix};
 \end{aligned}$$

1.3.2 α -curvatures

Proposition 1.4 The components of the α -curvature tensor $R_{ijkl}^{(\alpha)} = g_{bl}(\partial_{\alpha k} \Gamma_{jk}^b - \partial_{\alpha j} \Gamma_{ik}^b + \Gamma_{im}^b \Gamma_{jk}^m - \Gamma_{jm}^b \Gamma_{ik}^m)$, ($i, j, k, l = 1, 2, 3, 4$) are given by

$$\begin{aligned}
 \text{In[11]:= } \overset{\alpha}{R} &= \text{FullSimplify[Table[\sum_{b=1}^4 g[[b, l]] Simplify[-\frac{\partial \Gamma[[i, k]]}{\partial \alpha_j} + \frac{\partial \Gamma[[j, k]]}{\partial \alpha_i} +} \\
 &\quad \sum_{m=1}^4 \text{Simplify}[\Gamma[[i, m]] \Gamma[[j, k]]^m - \Gamma[[j, m]] \Gamma[[i, k]]^m], \\
 &\quad \{i, 4\}, \{j, 4\}, \{k, 4\}, \{l, 4\}]];
 \end{aligned}$$

Since the analytical expression for the curvature tensor is very large, we report the components $R_{ijkl}^{(\alpha)}$:

$$\begin{aligned}
 \text{In[12]:= } &\text{FullSimplify}[\{\overset{\alpha}{R}[[1, 2, 1, 2]], \overset{\alpha}{R}[[1, 2, 2, 3]], \overset{\alpha}{R}[[1, 4, 1, 4]], \\
 &\quad \overset{\alpha}{R}[[1, 4, 3, 4]], \overset{\alpha}{R}[[2, 3, 2, 3]], \overset{\alpha}{R}[[2, 4, 2, 4]], \overset{\alpha}{R}[[3, 4, 3, 4]]\}] \\
 \text{Out[12]= } &\left\{ \frac{(\alpha^2 - 1)\alpha_3^2}{4\alpha_1\alpha_2^2(\alpha_1 + \alpha_3)^3}, \frac{(\alpha^2 - 1)\alpha_3}{4\alpha_2^2(\alpha_1 + \alpha_3)^3}, \frac{(\alpha^2 - 1)\alpha_3}{4(\alpha_1 + \alpha_3)^3\alpha_4^2}, -\frac{(\alpha^2 - 1)\alpha_1}{4(\alpha_1 + \alpha_3)^3\alpha_4^2}, \right. \\
 &\quad \left. \frac{(\alpha^2 - 1)\alpha_1}{4\alpha_2^2(\alpha_1 + \alpha_3)^3}, \frac{(\alpha^2 - 1)\alpha_1\alpha_3}{4\alpha_2^2(\alpha_1 + \alpha_3)^2\alpha_4^2}, \frac{(\alpha^2 - 1)\alpha_1^2}{4\alpha_3(\alpha_1 + \alpha_3)^3\alpha_4^2} \right\}
 \end{aligned}$$

while the other independent components are zero.

Proposition 1.5 The α -sectional curvatures:

$$\varrho^{(\alpha)}(i, j) = -\frac{R_{ijij}^{(\alpha)}}{(g_{ii}g_{jj} - g_{ij}^2)}, \quad (i, j = 1, 2, 3, 4)$$

are given by

$$\begin{aligned}
 \text{In}[13] := & \alpha \varrho = \text{FullSimplify}[\text{Table}[\text{If}[i == j, 0, -\frac{\alpha R[i, j, i, j]}{g[i, i] g[j, j] - g[i, j]^2}], \{i, 4\}, \{j, 4\}]] \\
 \text{Out}[13] = & \begin{pmatrix} 0 & -\frac{(\alpha^2-1)\alpha_3}{4(\alpha_1+\alpha_3)} & 0 & -\frac{(\alpha^2-1)\alpha_3}{4(\alpha_1+\alpha_3)} \\ -\frac{(\alpha^2-1)\alpha_3}{4(\alpha_1+\alpha_3)} & 0 & -\frac{(\alpha^2-1)\alpha_1}{4(\alpha_1+\alpha_3)} & \frac{1}{4}(1-\alpha^2) \\ 0 & -\frac{(\alpha^2-1)\alpha_1}{4(\alpha_1+\alpha_3)} & 0 & -\frac{(\alpha^2-1)\alpha_1}{4(\alpha_1+\alpha_3)} \\ -\frac{(\alpha^2-1)\alpha_3}{4(\alpha_1+\alpha_3)} & \frac{1}{4}(1-\alpha^2) & -\frac{(\alpha^2-1)\alpha_1}{4(\alpha_1+\alpha_3)} & 0 \end{pmatrix}
 \end{aligned}$$

Proposition 1.6 The α -mean curvatures:

$$\varrho^{(\alpha)}(i) = \frac{1}{3} \sum_{j=1, j \neq i}^4 \varrho^{(\alpha)}(i, j), \quad (i = 1, 2, 3, 4)$$

are given by

$$\begin{aligned}
 \text{In}[14] := & \alpha \varrho = \text{FullSimplify}[\text{Table}[\sum_{j=1}^4 \frac{1}{3} \alpha \varrho[i, j], \{i, 4\}]] \\
 \text{Out}[14] = & \left\{ -\frac{(\alpha^2-1)\alpha_3}{6(\alpha_1+\alpha_3)}, \frac{1}{6}(1-\alpha^2), -\frac{(\alpha^2-1)\alpha_1}{6(\alpha_1+\alpha_3)}, \frac{1}{6}(1-\alpha^2) \right\}
 \end{aligned}$$

Contracting $R_{ijkl}^{(\alpha)}$ with g^{il} , we obtain the components $R_{jk}^{(\alpha)}$ of the Ricci tensor.

Proposition 1.7 The components of the α -Ricci tensor are given by the symmetric matrix $R^{(\alpha)} = [R_{jk}^{(\alpha)}]$.

$$\begin{aligned}
 \text{In}[15] := & \alpha \bar{R} = \text{FullSimplify}[\text{Table}[\sum_{l=1}^4 \sum_{i=1}^4 g^{-1}[i, l] \alpha R[i, j, k, l], \{j, 4\}, \{k, 4\}]] \\
 \text{Out}[15] = & \begin{pmatrix} -\frac{(\alpha^2-1)\alpha_3}{2\alpha_1(\alpha_1+\alpha_3)^2} & 0 & \frac{\alpha^2-1}{2(\alpha_1+\alpha_3)^2} & 0 \\ 0 & -\frac{(\alpha^2-1)\alpha_3}{2\alpha_2^2(\alpha_1+\alpha_3)} & 0 & 0 \\ \frac{\alpha^2-1}{2(\alpha_1+\alpha_3)^2} & 0 & -\frac{(\alpha^2-1)\alpha_1}{2\alpha_3(\alpha_1+\alpha_3)^2} & 0 \\ 0 & 0 & 0 & -\frac{(\alpha^2-1)\alpha_1}{2(\alpha_1+\alpha_3)\alpha_4^2} \end{pmatrix}
 \end{aligned}$$

The α -eigenvalues and the α -eigenvectors of the α -Ricci tensor are given by

$$\begin{aligned}
 \text{In}[16] := & \text{FullSimplify}[\text{Eigenvalues}[\alpha \bar{R}]] \\
 \text{Out}[16] = & \left\{ 0, -\frac{(\alpha^2-1)\alpha_3}{2\alpha_2^2(\alpha_1+\alpha_3)}, \frac{1}{2}(\alpha^2-1) \left(\frac{2}{(\alpha_1+\alpha_3)^2} - \frac{1}{\alpha_1\alpha_3} \right), -\frac{(\alpha^2-1)\alpha_1}{2(\alpha_1+\alpha_3)\alpha_4^2} \right\}
 \end{aligned}$$

In[17]:= FullSimplify[Eigenvalues[R]]

$$\text{Out[17]} = \begin{pmatrix} \frac{\alpha_1}{\alpha_3} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\alpha_3}{\alpha_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proposition 1.8 The manifold F has a constant α -scalar curvature $R^{(\alpha)} = g^{ij} R_{ij}^{(\alpha)}$.

In[18]:= $R = \text{FullSimplify}[\sum_{j=1}^4 \sum_{i=1}^4 g^{-1}[[i, j]] R[[i, j]]]$

$$\text{Out[18]} = -\frac{3}{2} (\alpha^2 - 1)$$

Note that the Freund manifold F has a positive constant scalar curvature $R^{(0)} = \frac{3}{2}$ when $\alpha = 0$, so geometrically it constitutes part of a sphere (compare, for example, Kobayashi and Nomizu [9]).

■ 2. Submanifolds of the Freund 4-Manifold F

We consider four submanifolds F_i ($i = 1, 2, 3, 4$) of the 4-manifold F of Freund bivariate exponential distributions (1) $f(x, y; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, which includes the case of statistically independent random variables. It also includes the special case of an Absolutely Continuous Bivariate Exponential Distribution called ACBED (or ACBVE) by Block and Basu (compare Hutchinson and Lai [10]). We use the coordinate system $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for the submanifolds F_i , ($i \neq 4$) and the coordinate system $(\lambda_1, \lambda_{12}, \lambda_2)$ for ACBED of the Block and Basu case.

□ 2.1. Submanifold $F_1 \subset F: \alpha_2 = \alpha_1$ and $\alpha_4 = \alpha_3$

The distributions are of the form:

$$f(x, y; \alpha_1, \alpha_3) = f_1(x; \alpha_1) f_2(y; \alpha_3), \quad (12)$$

where f_i are the density functions of the one-dimensional exponential distributions with the parameters $\alpha_i > 0$ ($i = 1, 3$). This is the case for statistical independence of X and Y , so the space F_1 is the direct product of two Riemannian spaces:

$$\{f_1(x; \alpha_1) : f_1(x; \alpha_1) = \alpha_1 e^{-\alpha_1 x}, \alpha_1 > 0\}$$

and

$$\{f_2(y; \alpha_3) : f_2(y; \alpha_3) = \alpha_3 e^{-\alpha_3 y}, \alpha_3 > 0\}.$$

Proposition 2.1 The Fisher information metric $[g_{ij}]$ is given by

$$[g_{ij}] = \begin{pmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_3^2} \end{pmatrix}. \quad (13)$$

Proposition 2.2 The components of $\nabla^{(\alpha)}$ are given by

$$\Gamma^1_{11,1} = \frac{\alpha-1}{\alpha_1^3}, \quad \Gamma^1_{22,2} = \frac{\alpha-1}{\alpha_3^3}, \quad \Gamma^1_{11} = \frac{\alpha-1}{\alpha_1}, \quad \Gamma^2_{22} = \frac{\alpha-1}{\alpha_3}, \quad (14)$$

while the other components are zero.

Proposition 2.3 The α -curvature tensor, α -Ricci tensor, and α -scalar curvature of F_1 are zero.

□ 2.2. Submanifold $F_2 \subset F: \alpha_3 = \alpha_1$ and $\alpha_4 = \alpha_2$

The distributions are of the form:

$$f(x, y) = \begin{cases} \alpha_1 \alpha_2 e^{-\alpha_2 y - (2\alpha_1 - \alpha_2)x} & \text{for } 0 < x < y \\ \alpha_1 \alpha_2 e^{-\alpha_2 x - (2\alpha_1 - \alpha_2)y} & \text{for } 0 < y < x \end{cases} \quad (15)$$

with parameters $\alpha_1, \alpha_2 > 0$. The covariance, correlation coefficient, and marginal functions of X and Y are given by

$$\text{Cov}(X, Y) = \frac{1}{4} \left(\frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2} \right), \quad (16)$$

$$\rho(X, Y) = 1 - \frac{4\alpha_1^2}{3\alpha_1^2 + \alpha_2^2}, \quad (17)$$

$$f_X(x) = \left(\frac{\alpha_1}{2\alpha_1 - \alpha_2} \right) \alpha_2 e^{-\alpha_2 x} + \left(\frac{\alpha_1 - \alpha_2}{2\alpha_1 - \alpha_2} \right) (2\alpha_1) e^{-2\alpha_1 x}, \quad x \geq 0 \quad (18)$$

$$f_X(x) = \left(\frac{\alpha_1}{2\alpha_1 - \alpha_2} \right) \alpha_2 e^{-\alpha_2 x} + \left(\frac{\alpha_1 - \alpha_2}{2\alpha_1 - \alpha_2} \right) (2\alpha_1) e^{-2\alpha_1 x}, \quad x \geq 0. \quad (19)$$

Note that $\rho(X, Y) = 0$ when $\alpha_1 = \alpha_2$.

F_2 forms an exponential family, with natural parameters (α_1, α_2) and the potential function

$$\ln[\eta] := \psi = -\log(\alpha_1) - \log(\alpha_2);$$

Proposition 2.4 The submanifold F_2 is an isometric diffeomorph of the submanifold F_1 .

Proof: Since ψ is a potential function, the Fisher information metric is given by the Hessian of ψ , that is,

$$g_{ij} = \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j}, \quad (i = 1, 2). \quad (20)$$

Then we have the Fisher information metric (10) by a straightforward calculation.

2.2.1 Mutually Dual Foliations

Since $\nabla_{\partial_i}^{(1)} \partial_j = 0$, (α_1, α_2) is a 1-affine coordinate system, and the (-1) -affine coordinate system is given by

$$\text{In}[20]:= \eta = \text{Outer}[\text{FullSimplify}[\frac{\partial \psi}{\partial \#1}, \{\alpha_1 > 0, \alpha_2 > 0\}] \&, \{\alpha_1, \alpha_2\}]$$

$$\text{Out}[20]:= \left\{ -\frac{1}{\alpha_1}, -\frac{1}{\alpha_2} \right\}$$

These coordinates have the potential function

$$\text{In}[21]:= \phi = \text{FullSimplify}[\sum_{i=1}^2 \eta[i] \alpha_i - \psi]$$

$$\text{Out}[21]:= \log(\alpha_1) + \log(\alpha_2) - 2$$

So the coordinates (α_1, α_2) and $(-\frac{1}{\alpha_1}, -\frac{1}{\alpha_2})$ are mutually dual with respect to the Fisher metric, and the tetrad $\{F_2, g, \nabla^{(1)}, \nabla^{(-1)}\}$ is a dually flat space. Therefore, F_2 has dually orthogonal foliations.

For example, take (α_1, η_2) as a coordinate system for F_2 ; then

$$f(x, y) = \begin{cases} -\frac{\alpha_1}{\eta_2} e^{(\frac{1}{\eta_2})y - (2\alpha_1 + \frac{1}{\eta_2})x} & \text{for } 0 < x < y \\ \frac{\alpha_1}{\eta_2} & \\ -\frac{\alpha_1}{\eta_2} e^{(\frac{1}{\eta_2})x - (2\alpha_1 + \frac{1}{\eta_2})y} & \text{for } 0 < y < x \end{cases} \quad (21)$$

and the Fisher information metric is

$$\begin{pmatrix} -\frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{(\eta_2)^2} \end{pmatrix}. \quad (22)$$

2.2.2 Neighbourhoods of Independence in F_2

An important practical application of the Freund submanifold F_2 is the representation of a bivariate stochastic process for which the marginals are identical exponentials. The next result is important because it provides a topological neighbourhood of that subspace W in F_2 consisting of the bivariate processes that have zero covariance: we obtain a neighbourhood of independence for random, that is, exponentially distributed processes.

Proposition 2.5 Let $\{F_2, g, \nabla^{(1)}, \nabla^{(-1)}\}$ be the manifold F_2 with Fisher metric g and exponential connection $\nabla^{(1)}$. Then F_2 can be realized in \mathbb{R}^3 by the graph of a potential function, namely, F_2 can be realized by the affine immersion $\{b, \xi\}$

$$b : F_2 \rightarrow \mathbb{R}^3 : \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \psi \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (23)$$

where $\psi = -\log(\alpha_1) - \log(\alpha_2)$ and ξ is the transversal vector field along b .

In F_2 , the submanifold W consisting of the independent case ($\alpha_1 = \alpha_2$) is represented by the curve

$$(0, \infty) \rightarrow \mathbb{R}^3 : (\alpha_1, \alpha_1) \mapsto (\alpha_1, \alpha_1, -2 \log(\alpha_1)), \quad \xi = (0, 0, 1). \quad (24)$$

This is illustrated in the graphic that shows S , an affine embedding of F_2 as a surface in \mathbb{R}^3 , and T , an \mathbb{R}^3 -tubular neighbourhood of W , as the curve $\alpha_1 = \alpha_2$ in the surface. This curve represents all bivariate distributions having identical exponential marginals and zero covariance; its tubular neighbourhood represents departures from independence. We parametrize here with $a = \alpha_1, b = \alpha_2$.

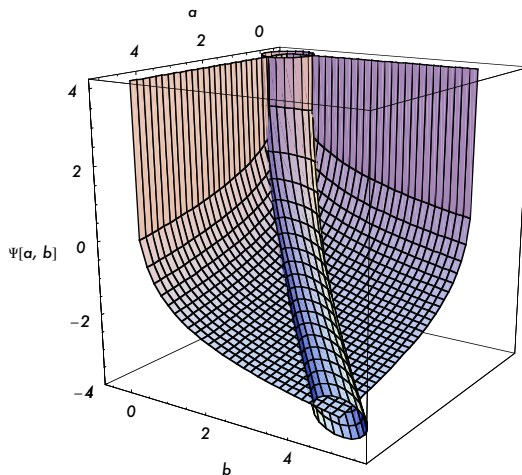
```
In[22]:= S = ParametricPlot3D[Evaluate[{a, b, -Log[a] - Log[b]}], {a, 0.00001, 5},
  {b, 0.00001, 5}, ViewPoint -> {3, 2, 1}, BoxRatios -> {1, 1, 1},
  AxesLabel -> {a, b, Psi(a, b)}, DisplayFunction -> Identity];
```

```
T = ParametricPlot3D[{a - (0.6 Cos[theta] / Sqrt[1 + a^2]), a - 0.6 Sin[theta],
```

```
  - (0.6 a Cos[theta] / Sqrt[1 + a^2]) - 2 Log[a]} // Evaluate,
```

```
{a, 0.00001, 5}, {theta, 0, 2 Pi}, PlotPoints -> {20, 20},
  ViewPoint -> {3, 2, 1}, BoxRatios -> {1, 1, 1},
  AxesLabel -> {a, b, Psi(a, b)}, DisplayFunction -> Identity];
```

```
In[24]:= Show[{S, T}, DisplayFunction -> $DisplayFunction]
```



□ **2.3. Submanifold $F_3 \subset F$: $\alpha_2 = \alpha_4 = \alpha_1 + \alpha_3$**

The distributions are of the form:

$$f(x, y; \alpha_1, \alpha_2) = \begin{cases} \alpha_1 (\alpha_1 + \alpha_3) e^{-(\alpha_1 + \alpha_3)y} & \text{for } 0 < x < y \\ \alpha_3 (\alpha_1 + \alpha_3) e^{-(\alpha_1 + \alpha_3)x} & \text{for } 0 < y < x \end{cases} \quad (25)$$

with parameters $\alpha_1, \alpha_3 > 0$. The covariance, correlation coefficient, and marginal functions of X and Y are given by

$$\text{Cov}(X, Y) = \frac{\alpha_1^2 + \alpha_1 \alpha_3 + \alpha_3^2}{(\alpha_1 + \alpha_3)^4}, \quad (26)$$

$$\rho(X, Y) = \frac{\alpha_1^2 + \alpha_1 \alpha_3 + \alpha_3^2}{\sqrt{2(\alpha_1 + \alpha_3)^2 - \alpha_1^2} \sqrt{2\alpha_1^2 + 4\alpha_1 \alpha_3 + \alpha_3^2}}, \quad (27)$$

$$f_X(x) = (\alpha_3 (\alpha_1 + \alpha_3)x + \alpha_1) e^{-(\alpha_1 + \alpha_3)x}, \quad x \geq 0 \quad (28)$$

$$f_Y(y) = (\alpha_1 (\alpha_1 + \alpha_3)y + \alpha_3) e^{-(\alpha_1 + \alpha_3)y}, \quad y \geq 0. \quad (29)$$

Note that the correlation coefficient is positive.

Proposition 2.6 The Fisher information metric $[g_{ij}]$ is given by

$$[g_{ij}] = \begin{pmatrix} \frac{2\alpha_1 + \alpha_3}{\alpha_1(\alpha_1 + \alpha_3)^2} & \frac{1}{(\alpha_1 + \alpha_3)^2} \\ \frac{1}{(\alpha_1 + \alpha_3)^2} & \frac{\alpha_1 + 2\alpha_3}{\alpha_3(\alpha_1 + \alpha_3)^2} \end{pmatrix}. \quad (30)$$

Proposition 2.7 The α -connection components of F_3 are given by

$$\begin{aligned} \Gamma_{11,1}^{(\alpha)} &= \frac{4(\alpha - 1)\alpha_1^2 + (\alpha - 3)\alpha_3\alpha_1 - (\alpha + 1)\alpha_3^2}{2\alpha_1^2(\alpha_1 + \alpha_3)^3}, \\ \Gamma_{11}^1 &= \frac{(\alpha_1 + 2\alpha_3)(4(\alpha - 1)\alpha_1^2 + (\alpha - 3)\alpha_3\alpha_1 - (\alpha + 1)\alpha_3^2)}{4\alpha_1(\alpha_1 + \alpha_3)^3}, \\ \Gamma_{11}^2 &= \frac{\alpha_3(-4(\alpha - 1)\alpha_1^2 - (\alpha - 3)\alpha_3\alpha_1 + (\alpha + 1)\alpha_3^2)}{4\alpha_1(\alpha_1 + \alpha_3)^3}, \end{aligned} \quad (31)$$

while the other independent components are zero.

Proposition 2.8 The α -curvature tensor, α -Ricci tensor, and α -scalar curvature of F_3 are zero.

□ 2.4. Submanifold $F_4 \subset F$: ACBED of Block and Basu

The distributions are of the form:

$$f(x, y; \lambda_1, \lambda_{12}, \lambda_2) = \begin{cases} \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} e^{-\lambda_1 x - (\lambda_2 + \lambda_{12}) y} & \text{for } 0 < x < y \\ \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_{12}) x - \lambda_2 y} & \text{for } 0 < y < x, \end{cases} \quad (32)$$

where the parameters $\lambda_1, \lambda_{12}, \lambda_2$ are positive, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. This distribution was derived originally by omitting the singular part of Marshall and Olkin's distribution (compare [10]); Block and Basu called it the ACBED to emphasize that it is an absolutely continuous part of a bivariate exponential distribution. Alternatively, it can be derived from the Freund distribution (1), with

$$\begin{aligned} \alpha_1 &= \lambda_1 + \frac{\lambda_1 \lambda_{12}}{\lambda_1 + \lambda_2}, \\ \alpha_2 &= \lambda_1 + \lambda_{12}, \\ \alpha_3 &= \lambda_2 + \frac{\lambda_2 \lambda_{12}}{\lambda_1 + \lambda_2}, \\ \alpha_4 &= \lambda_2 + \lambda_{12}. \end{aligned}$$

By substitution we obtained the covariance, correlation coefficient, and marginal functions of X and Y :

$$\text{Cov}(X, Y) = \frac{(\lambda_1 + \lambda_2)^2 (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12}) - \lambda^2 \lambda_1 \lambda_2}{\lambda^2 (\lambda_1 + \lambda_2)^2 (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})}, \quad (33)$$

$$\begin{aligned} \rho(X, Y) &= ((\lambda_1 + \lambda_2)^2 (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12}) - \lambda^2 \lambda_1 \lambda_2) / \\ &\quad \sqrt{(((\lambda_1 + \lambda_2)^2 (\lambda_2 + \lambda_{12})^2 + \lambda_1 \lambda^2 (\lambda_1 + 2 \lambda_2)) \\ &\quad ((\lambda_1 + \lambda_2)^2 (\lambda_1 + \lambda_{12})^2 + \lambda_2 \lambda^2 (\lambda_2 + 2 \lambda_1))),} \end{aligned} \quad (34)$$

$$f_X(x) = \left(\frac{-\lambda_{12}}{\lambda_1 + \lambda_2} \right) \lambda e^{-\lambda x} + \left(\frac{\lambda}{\lambda_1 + \lambda_2} \right) (\lambda_1 + \lambda_{12}) e^{-(\lambda_1 + \lambda_{12}) x}, \quad x \geq 0 \quad (35)$$

$$f_Y(y) = \left(\frac{-\lambda_{12}}{\lambda_1 + \lambda_2} \right) \lambda e^{-\lambda y} + \left(\frac{\lambda}{\lambda_1 + \lambda_2} \right) (\lambda_2 + \lambda_{12}) e^{-(\lambda_2 + \lambda_{12}) y}, \quad y \geq 0. \quad (36)$$

Note that the correlation coefficient is positive, and the marginal functions are a negative mixture of two exponentials.

The Christoffel symbols, curvature tensor, Ricci tensor, sectional curvatures, mean curvatures, and scalar curvature were computed, but are not listed because they are somewhat cumbersome.

When $\lambda_2 = \lambda_1$, this family of distributions becomes an exponential family with natural parameters $(\lambda_1, \lambda_{12})$ and potential function $\psi = \log(2) - \log(\lambda_1 + \lambda_{12}) - \log(2\lambda_1 + \lambda_{12})$, so it would be easy to derive the α -geometry, for example,

$$[g_{ij}] = \begin{pmatrix} \frac{4}{(2\lambda_1 + \lambda_{12})^2} + \frac{1}{(\lambda_1 + \lambda_{12})^2} & \frac{2}{(2\lambda_1 + \lambda_{12})^2} + \frac{1}{(\lambda_1 + \lambda_{12})^2} \\ \frac{2}{(2\lambda_1 + \lambda_{12})^2} + \frac{1}{(\lambda_1 + \lambda_{12})^2} & \frac{1}{(2\lambda_1 + \lambda_{12})^2} + \frac{1}{(\lambda_1 + \lambda_{12})^2} \end{pmatrix}, \quad (37)$$

and the $\nabla^{(\alpha)}$ components are as follows:

$${}^1\Gamma = \begin{pmatrix} \frac{1-\alpha}{\lambda_1 + \lambda_{12}} + \frac{4(\alpha-1)}{2\lambda_1 + \lambda_{12}} & \frac{(\alpha-1)\lambda_{12}}{(\lambda_1 + \lambda_{12})(2\lambda_1 + \lambda_{12})} \\ \frac{(\alpha-1)\lambda_{12}}{(\lambda_1 + \lambda_{12})(2\lambda_1 + \lambda_{12})} & -\frac{(\alpha-1)\lambda_1}{(\lambda_1 + \lambda_{12})(2\lambda_1 + \lambda_{12})} \end{pmatrix}, \quad (38)$$

$${}^2\Gamma = \begin{pmatrix} -\frac{2(\alpha-1)\lambda_{12}}{(\lambda_1 + \lambda_{12})(2\lambda_1 + \lambda_{12})} & \frac{2(\alpha-1)\lambda_1}{(\lambda_1 + \lambda_{12})(2\lambda_1 + \lambda_{12})} \\ \frac{2(\alpha-1)\lambda_1}{(\lambda_1 + \lambda_{12})(2\lambda_1 + \lambda_{12})} & \frac{1-\alpha}{2\lambda_1 + \lambda_{12}} + \frac{2(\alpha-1)}{\lambda_1 + \lambda_{12}} \end{pmatrix}, \quad (39)$$

while the α -curvatures vanish.

■ 3. Concluding Remarks

We have derived the information geometry of the 4-manifold F of Freund bivariate mixture exponential distributions, which admits positive and negative covariance. The α -curvature objects are derived for F and those on four submanifolds, including the case of statistically independent random variables and the special case ACBED of Block and Basu. We provide examples of neighbourhoods of the independent case for bivariate distributions having identical exponential marginals and zero covariance. The Freund manifold has a constant 0-scalar curvature, so geometrically it constitutes part of a sphere.

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