On the Perimeter of an Ellipse

Paul Abbott

Computing accurate approximations to the perimeter of an ellipse is a favorite problem of mathematicians, attracting luminaries such as Ramanujan [1, 2, 3]. As is well known, the perimeter $\mathcal{P}$ of an ellipse with semimajor axis a and semiminor axis b can be expressed exactly as a complete elliptic integral of the second kind.

What is less well known is that the various exact forms attributed to Maclau rin, Gauss-Kummer, and Euler are related via quadratic hypergeometric transformations. These transformations lead to additional identities, including a particularly elegant formula symmetric in a and b.

Approximate formulas can, of course, be obtained by truncating the series representations of exact formulas. For example, Kepler used the geometric mean, $\mathcal{P} \approx 2\pi\sqrt{a\,b}$, as a lower bound for the perimeter. In this article, we examine the properties of a number of approximate formulas, using series methods, polynomial interpolation, rational polynomial approximants, and minimax methods.

■ Introduction

The well-known formula for the perimeter $\mathcal{P}$ of an ellipse with semimajor axis a and semiminor axis b can be expressed exactly as a complete elliptic integral of the second kind, which can also be written as a Gaussian hypergeometric function,

$$\mathcal{P} = 4\,a\,E\left(1 - \frac{b^2}{a^2}\right) = 2\,\pi\,a\,{}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{b^2}{a^2}\right).$$

The quadratic hypergeometric transformations [4, 5] lead to additional identities, including a particularly elegant formula, symmetric in a and b,

$$\mathcal{P} = 2\,\pi\sqrt{a\,b}\;P_{\frac{1}{2}}\left(\frac{a^2 + b^2}{2\,a\,b}\right),$$

where $P_\nu(z)$ is a Legendre function.

Cartesian Equation

The Cartesian equation for an ellipse with center at (0, 0), semimajor axis a, and semiminor axis b reads

In[1]:= $\mathcal{E}(\text{x\_}, \text{y\_}) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 == 1;$

Introducing the parameter φ into the Cartesian coordinates, as $(x = a\sin(\varphi), y = b\cos(\varphi))$, we verify that the ellipse equation is satisfied.

In[2]:= **Simplify[$\mathcal{E}(a\sin(\varphi),\ b\cos(\varphi))$]**

Out[2]= True

Arclength

In general, the parametric arclength is defined by

$$\mathcal{L} = \int_{\varphi_1}^{\varphi_2} \sqrt{\left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2}\, d\varphi. \tag{1}$$

The arclength of an ellipse as a function of the parameter φ is an (incomplete) elliptic integral of the second kind.

In[3]:= $\mathcal{L}(\varphi\_) = \text{With}\Big[\{x = a\sin(\varphi),\ y = b\cos(\varphi)\},$

$\text{Simplify}\Big[\int \sqrt{\left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2}\, d\varphi,\ a > b > 0 \bigwedge 0 < \varphi < \frac{\pi}{2}\Big]\Big]$

Out[3]= $a\, E\left(\varphi \,\middle|\, 1 - \frac{b^2}{a^2}\right)$

Since

In[4]:= $\mathcal{L}(0) == 0$

Out[4]= True

the arclength of the ellipse is

$$\mathcal{L}(\varphi) = a\, E(\varphi \mid e^2), \tag{2}$$

where the eccentricity e is defined by

In[5]:= $e(\text{a\_}, \text{b\_}) = \sqrt{1 - \frac{b^2}{a^2}};$

■ Perimeter

Since the parameter ranges over $0 \le \varphi \le \frac{\pi}{2}$ for one quarter of the ellipse, the perimeter of the ellipse is

In[6]:= $\mathcal{P}_1(\text{a\_}, \text{b\_}) = 4\,\mathcal{L}\left(\frac{\pi}{2}\right)$

Out[6]= $4\,a\,E\left(1 - \frac{b^2}{a^2}\right)$

That is, $\mathcal{P} = 4\,a\,E(e^2)$, where $E(m)$ is the complete elliptic integral of the second kind.

□ Alternative Expressions for the Perimeter

The given expression for the perimeter of the ellipse is *unsymmetrical* with respect to the parameters a and b. This is "unphysical" in that both parameters, being lengths of the (major and minor) axes, should be on the same footing. We can expect that a *symmetric* formula, when truncated, will more accurately approximate the perimeter for both $a \ge b$ and $a \le b$.

Noting that the complete elliptic integral is a Gaussian hypergeometric function,

In[7]:= $_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; z\right)$

Out[7]= $\frac{2\,E(z)}{\pi}$

we obtain Maclaurin's 1742 formula [2]

In[8]:= $\mathcal{P}_1(a, b) == 2\,\pi\,a\,{_2F_1}\left(\frac{1}{2}, -\frac{1}{2}; 1; e(a, b)^2\right)$

Out[8]= True

Equivalent alternative expressions for the perimeter of the ellipse can be obtained from quadratic transformation formulas for Gaussian hypergeometric functions. For example, using functions.wolfram.com/07.23.17.0106.01, In[9]:= $$\text{Simplify}\left[{}_2F_1(\alpha, \beta; 2\beta; z) = \frac{{}_2F_1\left(\alpha, \alpha-\beta+\frac{1}{2}; \beta+\frac{1}{2}; \left(\frac{1-\sqrt{1-z}}{\sqrt{1-z}+1}\right)^2\right)}{\left(\frac{1}{2}\left(\sqrt{1-z}+1\right)\right)^{2\alpha}} \;/.\; \left\{\beta \to \frac{1}{2}, \alpha \to -\frac{1}{2}, z \to e(a, b)^2\right\}, a > b > 0\right]$$

Out[9]= $$4\, a\, E\left(1 - \frac{b^2}{a^2}\right) = (a+b)\,\pi\, {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(a-b)^2}{(a+b)^2}\right)$$

and noting that

In[10]:= $$\text{Simplify}\left[\frac{(a-b)^2}{(a+b)^2} = 1 - \frac{4\, a\, b}{(a+b)^2}\right]$$

Out[10]= True

we obtain the following symmetric formula

In[11]:= $$\mathcal{P}_2(\text{a\_}, \text{b\_}) = \pi\,(a+b)\, {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1 - \frac{4\, a\, b}{(a+b)^2}\right);$$

first obtained by Ivory in 1796, but known as the Gauss-Kummer series [2].

Introducing the homogeneous symmetric parameter $h = \frac{(a-b)^2}{(a+b)^2} = 1 - \frac{4\,a\,b}{(a+b)^2}$, we have (cf. mathworld.wolfram.com/Ellipse.html)

In[12]:= $$\text{Simplify}\left[\text{FunctionExpand}\left[\pi\,(a+b)\, {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; h\right)\right]\right]$$

Out[12]= $$2\,(a+b)\,(2\, E(h) + (h-1)\, K(h))$$

Explicitly, the Gauss-Kummer series reads

In[13]:= $$\mathcal{P}_3(\text{a\_}, \text{b\_}) = \text{FullSimplify}\left[\text{FunctionExpand}[\mathcal{P}_2(a, b)], a > b > 0\right]$$

Out[13]= $$4\,(a+b)\, E\left(1 - \frac{4\, a\, b}{(a+b)^2}\right) - \frac{8\, a\, b\, K\left(1 - \frac{4\,a\,b}{(a+b)^2}\right)}{a+b}$$

Instead, using functions.wolfram.com/07.23.17.0103.01, we obtain Euler's 1773 formula (see also [2])

In[14]:= $\text{Simplify}\left[{}_2F_1(\alpha,\ \beta;\ 2\,\beta;\ z) = \frac{{}_2F_1\left(\frac{\alpha}{2},\ \frac{\alpha+1}{2};\ \beta+\frac{1}{2};\ \frac{z^2}{(2-z)^2}\right)}{\left(1-\frac{z}{2}\right)^{\alpha}}\ /.\ \left\{\beta \to \frac{1}{2},\ \alpha \to -\frac{1}{2},\ z \to e(a,\ b)^2\right\}\right]$

Out[14]= $$4\,E\left(1-\frac{b^2}{a^2}\right) = \sqrt{\frac{2\,b^2}{a^2}+2}\ \pi\ {}_2F_1\left(-\frac{1}{4},\ \frac{1}{4};\ 1;\ \frac{\left(a^2-b^2\right)^2}{\left(a^2+b^2\right)^2}\right)$$

The hidden symmetry with respect to the interchange $a \leftrightarrow b$ is revealed.

In[15]:= **FullSimplify[%, $b > a > 0$]**

Out[15]= $$b\,E\left(1-\frac{a^2}{b^2}\right) = a\,E\left(1-\frac{b^2}{a^2}\right)$$

Defining

In[16]:= $$\mathcal{P}_4(\text{a\_},\ \text{b\_}) = \pi\sqrt{2\left(a^2+b^2\right)}\ {}_2F_1\left(\frac{1}{4},\ -\frac{1}{4};\ 1;\ \left(\frac{a^2-b^2}{a^2+b^2}\right)^2\right);$$

we can directly check the formula.

In[17]:= $\text{Simplify}[\text{FunctionExpand}[\mathcal{P}_4(a,\ b) = \mathcal{P}_1(a,\ b)],\ a > b > 0]$

Out[17]= True

□ Other Identities

There are many other possible transformation formulas that can be applied to obtain alternative expressions for the perimeter. For example, using functions.wolfram.com/07.23.17.0054.01 we obtain the following formula

In[18]:= $\mathcal{P}_5(\text{a\_},\ \text{b\_}) =$
$\mathcal{P}_2(a,\ b)\ /.\ {}_2F_1(\text{a\_},\ \text{b\_};\ \text{c\_};\ \text{z\_}) \to (1-z)^{-a-b+c}\ {}_2F_1(c-a,\ c-b;\ c;\ z)$

Out[18]= $$\frac{16\,a^2\,b^2\,\pi\ {}_2F_1\left(\frac{3}{2},\ \frac{3}{2};\ 1;\ 1-\frac{4\,a\,b}{(a+b)^2}\right)}{(a+b)^3}$$

The perimeter can also be expressed in terms of Legendre functions (see Sections 8.13 and 15.4 of [6]). For example, using 15.4.15 of [6] we obtain the elegant and simple symmetric formula

In[19]:= $\mathcal{P}_6$(a_, b_) =

$$\text{Simplify}\Big[\mathcal{P}_2(a, b) \;/.\; {}_2F_1(\text{a\_}, \text{b\_}; \text{c\_}; \text{x\_}) :\to \Gamma(a - b + 1)\,(1 - x)^{-b}\,(-x)^{\frac{b-a}{2}}\, P_{-b}^{b-a}\!\left(\frac{x+1}{1-x}\right) \;/;\; c = a - b + 1,\ a > 0 \land b > 0\Big]$$

Out[19]= $2\sqrt{a\,b}\;\pi\,P_{\frac{1}{2}}\!\left(\frac{a^2+b^2}{2\,a\,b}\right)$

Alternatively, this result follows directly from 8.13.6 of [6] with $e^{\eta} = \frac{a}{b} \Longrightarrow \cosh(\eta) = \frac{a^2+b^2}{2\,a\,b}$. This form can be used to prove that the perimeter of an ellipse is a homogenous mean (cf. [7]), extending the arithmetic-geometric mean (AGM) already used as a tool for computing elliptic integrals [8].

Using functions.wolfram.com/07.07.26.0001.01 gives yet another formula involving complete elliptic integrals.

In[20]:= $\mathcal{P}_7$(a_, b_) =

$$\text{Simplify}\Big[\text{FunctionExpand}\Big[\mathcal{P}_6(a, b) \;/.\; P_{\nu\_}(\text{z\_}) \to {}_2F_1\Big(-\nu, \nu + 1; 1; \frac{1-z}{2}\Big)\Big]\Big]$$

Out[20]= $4\sqrt{a\,b}\left(2\,E\!\left(-\frac{(a-b)^2}{4\,a\,b}\right) - K\!\left(-\frac{(a-b)^2}{4\,a\,b}\right)\right)$

□ Comparisons

Here we compare the seven formulas for $b = 2\,a$,

In[21]:= **Simplify[{$\mathcal{P}_1(a, 2\,a)$, $\mathcal{P}_2(a, 2\,a)$, $\mathcal{P}_3(a, 2\,a)$, $\mathcal{P}_4(a, 2\,a)$, $\mathcal{P}_5(a, 2\,a)$, $\mathcal{P}_6(a, 2\,a)$, $\mathcal{P}_7(a, 2\,a)$}, $a > 0$]**

Out[21]= $\Big\{4\,a\,E(-3),\ 3\,a\,\pi\,{}_2F_1\!\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{9}\right),\ \frac{4}{3}\,a\left(9\,E\!\left(\frac{1}{9}\right) - 4\,K\!\left(\frac{1}{9}\right)\right),$
$\sqrt{10}\;a\,\pi\,{}_2F_1\!\left(\frac{1}{4}, -\frac{1}{4}; 1; \frac{9}{25}\right),\ \frac{64}{27}\,a\,\pi\,{}_2F_1\!\left(\frac{3}{2}, \frac{3}{2}; 1; \frac{1}{9}\right),$
$2\sqrt{2}\;a\,\pi\,P_{\frac{1}{2}}\!\left(\frac{5}{4}\right),\ 4\sqrt{2}\;a\left(2\,E\!\left(-\frac{1}{8}\right) - K\!\left(-\frac{1}{8}\right)\right)\Big\}$

In[22]:= **N[%]**

Out[22]= {9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a, 9.68845 a}

In[23]:= **Equal @@ %**

Out[23]= True

and for $b = \frac{a}{3}$.

In[24]:= $\textbf{Simplify}\left[\left\{\mathcal{P}_1\left(a, \frac{a}{3}\right), \mathcal{P}_2\left(a, \frac{a}{3}\right), \mathcal{P}_3\left(a, \frac{a}{3}\right), \mathcal{P}_4\left(a, \frac{a}{3}\right), \mathcal{P}_5\left(a, \frac{a}{3}\right), \mathcal{P}_6\left(a, \frac{a}{3}\right), \mathcal{P}_7\left(a, \frac{a}{3}\right)\right\}, a > 0\right]$

Out[24]= $$\left\{4\, a\, E\left(\frac{8}{9}\right), \frac{4}{3}\, a\, \pi\, {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{4}\right), \frac{2}{3}\, a \left(8\, E\left(\frac{1}{4}\right) - 3\, K\left(\frac{1}{4}\right)\right), \frac{2}{3}\sqrt{5}\, a\, \pi\, {}_2F_1\left(\frac{1}{4}, -\frac{1}{4}; 1; \frac{16}{25}\right), \frac{3}{4}\, a\, \pi\, {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 1; \frac{1}{4}\right), \frac{2\, a\, \pi\, P_{\frac{1}{2}}\left(\frac{5}{3}\right)}{\sqrt{3}}, \frac{a\left(8\, E\left(-\frac{1}{3}\right) - 4\, K\left(-\frac{1}{3}\right)\right)}{\sqrt{3}}\right\}$$

In[25]:= **N[%]**

Out[25]= {4.45496 *a*, 4.45496 *a*, 4.45496 *a*, 4.45496 *a*, 4.45496 *a*, 4.45496 *a*, 4.45496 *a*}

In[26]:= **Equal @@ %**

Out[26]= True

■ Numerical Approximation

At www.ebyte.it/library/docs/math05a/EllipsePerimeterApprox05.html [1] we are encouraged to search for "…an efficient formula using only the four algebraic operations (if possible, avoiding even square-root) with a maximum error below 10 ppm. It would also be nice if such a formula were exact for both the circle and the degenerate flat ellipse".

The Gauss-Kummer series expressed as a function of the homogeneous variable $h = 1 - \frac{4\,a\,b}{(a+b)^2}$, reads

In[27]:= $\textbf{GaussKummer[h\_]} = \frac{\mathcal{P}_2(a, b)}{a + b} \;/.\; a + b \to \frac{2\sqrt{a\,b}}{\sqrt{1 - h}}$

Out[27]= $\pi\, {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; h\right)$

□ Series Expansions

The series expansion about $h = 0$ is useful for small h.

In[28]:= **GaussKummer[h] + O(h)^9**

Out[28]= $\pi + \frac{\pi h}{4} + \frac{\pi h^2}{64} + \frac{\pi h^3}{256} + \frac{25 \pi h^4}{16384} + \frac{49 \pi h^5}{65536} + \frac{441 \pi h^6}{1048576} + \frac{1089 \pi h^7}{4194304} + \frac{184041 \pi h^8}{1073741824} + O\left(h^9\right)$

Around $h = 1$, terms in $\log(1 - h)$ arise.

In[29]:= **Simplify[Series[GaussKummer[h], {h, 1, 2}], 0 < h < 1]**

Out[29]= $4 + (h-1) + \frac{1}{16}\left(-2\log(1-h) - 4\,\psi^{(0)}\left(\frac{3}{2}\right) - 4\gamma + 3\right)(h-1)^2 + O\left((h-1)^3\right)$

Using functions.wolfram.com/07.23.06.0015.0, we obtain the general term of this series (c.f. 17.3.33 through 17.3.36 of [6]),

In[30]:= **Simplify[GaussKummer[h] /.**

$$\boldsymbol{{}_2F_1(\text{a\_}, \text{b\_}; \text{c\_}; \text{z\_}) \rightarrow \text{With}\Bigg[\{n = c - a - b\}, \frac{(n-1)!\,\Gamma(a+b+n)}{\Gamma(a+n)\,\Gamma(b+n)} \sum_{k=0}^{n-1} \frac{(a)_k\,(b)_k\,(1-z)^k}{k!\,(1-n)_k} + \frac{\Gamma(a+b+n)}{\Gamma(a)\,\Gamma(b)} \left(\sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!} ((a+n)_k\,(b+n)_k)\left(-\log(1-z) + \psi(k+1) + \psi(k+n+1) - \psi(a+k+n) - \psi(b+k+n)\right)(1-z)^k\right)(z-1)^n\Bigg]\Bigg]}$$

Out[30]= $\frac{1}{4}\left(\left(\sum_{k=0}^{\infty} \frac{(1-h)^k \left(\frac{3}{2}\right)_k^2 \left(-\log(1-h) + \psi^{(0)}(k+1) + \psi^{(0)}(k+3) - 2\,\psi^{(0)}\left(k+\frac{3}{2}\right)\right)}{k!\,(k+2)!}\right)(h-1)^2 + 4\,(h+3)\right)$

□ Polynomial Approximants

Linear Approximant

From the exact values at $h = 0$,

In[31]:= **GaussKummer[0]**

Out[31]= π

and at $h = 1$,

In[32]:= **GaussKummer[1]**

Out[32]= 4

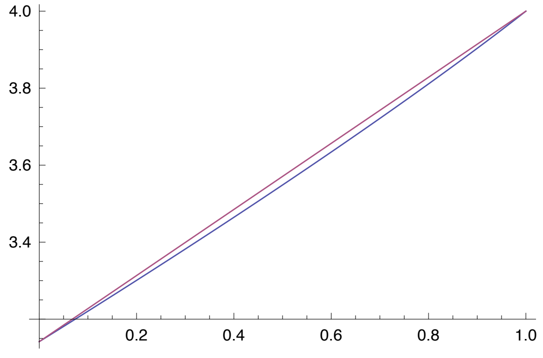
we construct the linear *extreme perfect* approximant.

In[33]:= **Linear[h_] = Simplify[(1 − *h*) GaussKummer[0] + *h* GaussKummer[1]]**

Out[33]= $\pi - h\,(-4 + \pi)$

In[34]:= **Plot[{GaussKummer[*h*], Linear[*h*]}, {*h*, 0, 1}]**

Out[34]=



Quadratic Approximant

The quadratic approximant, exact at $h = 0, \frac{1}{2}, 1$,

In[35]:= **FullSimplify$\left[\text{Table}\left[\{h, \text{GaussKummer}[h]\}, \left\{h, 0, 1, \frac{1}{2}\right\}\right]\right]$**

Out[35]= $\begin{pmatrix} 0 & \pi \\ \frac{1}{2} & \frac{2\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\,\Gamma\left(\frac{5}{4}\right)^2\right)}{\sqrt{\pi}} \\ 1 & 4 \end{pmatrix}$

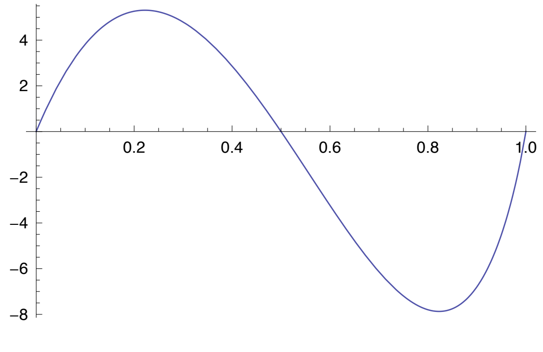
In[36]:= **Quadratic[h_] = *N*[InterpolatingPolynomial[%, *h*]]**

Out[36]= $(0.0891819\,(h - 0.5) + 0.813816)\,h + 3.14159$

has a maximum absolute relative error less than 8×10^{-4}.

In[37]:= **Plot$\left[10^4 \left(1 - \frac{\text{Quadratic}[h]}{\text{GaussKummer}[h]}\right), \{h, 0, 1\}\right]$**

Out[37]=



n^th^-order Polynomial Approximant

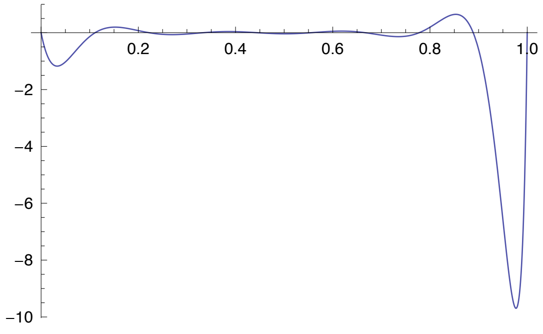
Here is the n^{th}-order "even-tempered" polynomial approximant, exact at $b = \frac{m}{n}$ for $m = 0, 1, \ldots, n$.

In[38]:=
```
poly[n_] := poly[n] = Function[b, Evaluate[InterpolatingPolynomial[
    N[Table[{b, GaussKummer[b]}, {b, 0, 1, 1/n}]], b]]]
```

The 9^{th}-order approximant has a maximum absolute relative error less than 10^{-5}.

In[39]:=
```
Plot[10^6 (1 - poly[9][b]/GaussKummer[b]),
  {b, 0, 1}, PlotRange → All, PlotPoints → 30]
```

Out[39]=



Chebyshev Polynomial Approximant

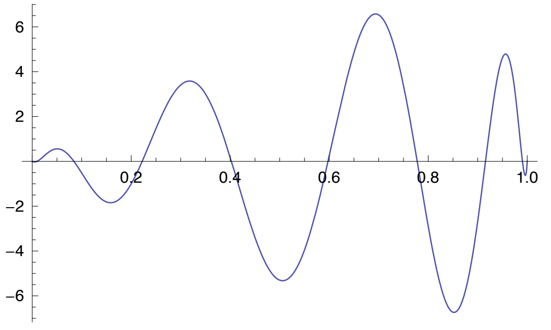
Sampling the Gauss-Kummer function at the zeros of $T_n(2x-1)$, which are at $x_m = \cos^2\left(\left(m+\frac{1}{4}\right)\frac{\pi}{n}\right)$, yields a Chebyshev polynomial approximant.

In[40]:=
```
Chebyshevpoly[n_] :=
  Chebyshevpoly[n] = Function[b, Evaluate[InterpolatingPolynomial[
     N[Join[(0  GaussKummer[0]
             1  GaussKummer[1]), Table[{cos²((m + 1/4) π / n),
         GaussKummer[cos²((m + 1/4) π / n)]}, {m, n}]]], b]]]
```

The 8^{th}-order approximant has a maximum absolute relative error less than 7×10^{-6}.

In[41]:= Plot$\left[10^6\left(1-\frac{\text{Chebyshevpoly[8][}h\text{]}}{\text{GaussKummer[}h\text{]}}\right)\right.$, {h, 0, 1}, PlotRange → All$\left.\right]$

Out[41]=



□ Rational Approximation

After loading the Function Approximations Package,

In[42]:= **<< FunctionApproximations`**

we obtain a family of $[N, M]$ rational polynomial minimax approximations.

In[43]:= **GKapprox[n_, m_] := GKapprox[*n*, *m*] = Function[*h*, Evaluate[**
MiniMaxApproximation[GaussKummer[*h*], {*h*, {0, 1}, *n*, *m*}]⟦2, 1⟧]]

For example, the [4, 3] minimax approximation,

In[44]:= **GKapprox[4, 3][*h*]**

Out[44]=

$$\frac{-0.0811183\,h^4+0.273498\,h^3+1.77163\,h^2-5.0554\,h+3.14159}{-0.14146\,h^3+1.01321\,h^2-1.8592\,h+1}$$

has (absolute) relative error at most 2.3×10^{-7}, but is not "extreme perfect".

In[45]:= Plot$\left[10^7\left(1-\frac{\text{GKapprox[4, 3][}h\text{]}}{\text{GaussKummer[}h\text{]}}\right),\ \{h, 0, 1\}\right]$

Out[45]=