Elliptic Rational Functions

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This article introduces a new algorithm for computing the elliptic rational function \( R_n(\xi, x) \). The key feature of the algorithm is a symbolic computation of some of the \( R_n(\xi, x) \) using only elementary functions. Our algorithm is more efficient than traditional methods based on the Jacobi elliptic functions and the complete elliptic integral, which can be numerically intensive. \( R_n(\xi, x) \) is used extensively in signal processing and system design as the best minimax approximation of a unit square pulse.

Introduction

Designers of many practical systems searched for a rational function \( R_n(\xi, x) \) in the variable \( x \) that has

- the equiripple property and \( | R_n(\xi, x) | \leq 1 \) over the interval \( |x| \leq 1 \)
- the largest value of \( \min( | R_n(\xi, x) | ) \) for \( |x| \geq \xi > 1 \)
- the minimal order \( n \)

The rational function with those properties was found [1, 2] by using the Jacobi elliptic functions [3] and it is referred to as the elliptic rational function [4].

A function has the equiripple property if it oscillates between maximums and minimums of equal amplitude [1]. A quotient of two polynomials is called a rational function in the variable \( x \), and the highest power in the polynomials is called the order of the rational function. The minimal value of \( | R_n(\xi, x) | \) for \( |x| \geq \xi \) is called the discrimination factor and is designated by \( L_n(\xi) \). In signal processing theory \( \xi \) is known as the selectivity factor and can be any real number greater than 1, \( \xi > 1 \). Note that \( R_n(\xi, x) \) is not a rational function in \( \xi \). A typical plot of \( R_n(\xi, x) \) is shown for \( n = 6 \) and \( \xi \approx 1.2 \).

We have defined the function EllipticRationalFunction that implements \( R_n(\xi, x) \). The algorithm is detailed in the subsequent sections.
From In[1]:= With[{n = 6, \(\xi = 1.2\), R = EllipticRationalFunction[n, \(\xi\), x]}, 
L = DiscriminationFactor[n, \(\xi\)];
a = Plot[EllipticRationalFunction[n, \(\xi\), x], {x, -1, 1}, 
  PlotStyle -> {Thickness[0.005]}, AxesLabel -> {x, R}, 
  Ticks -> {{-1, 1}, {-1, 1}}, DisplayFunction -> Identity];
b = Plot[Abs[EllipticRationalFunction[n, \(\xi\), x]], 
  {x, 1, 5 \(\xi\)}, PlotStyle -> {Thickness[0.005]}, 
  AxesLabel -> {x, Abs[R]}, PlotRange -> {Automatic, {0, 4000}}, 
  Ticks -> {Automatic, {Round[L], 3000}}, 
  GridLines -> {{\(\xi\), \{L\}}, DisplayFunction -> Identity};
Show[GraphicsArray[{a, b}]]

We used our symbolic algorithm for the elliptic rational function to optimize the 
symbolic performance of analog and digital systems. This optimization is not 
possible using traditional numeric algorithms. We derived closed-form formulas 
for designing high-speed low-consumption systems known as quadrature mirror 
filter banks [5]. \(R_n(\xi, x)\) is extensively used in analog signal processing as the 
best approximation function [6].

We found a new function, known as Minimum-Q Elliptic [4, 7], by symbolically 
optimizing the elliptic rational function. Minimum-Q Elliptic became a standard 
function in manufacturing integrated filters [7]. In addition, again using symbolic 
optimization, we implemented a very efficient digital signal processing (DSP) 
system using programmable logic devices and very large-scale integrated circuits 
[5, 8]. By an efficient DSP system, we mean processing by multiplierless systems 
that consist of a small number ofadders and binary shifters.

### Application

The quintessence of the importance of \(R_n(\xi, x)\) can be illustrated by the fact that 
\(R_n(\xi, x)\) can be used for generating the best minimax approximation of a unit 
square pulse [1, 4]:

\[
\frac{1}{1 + e^{2 R_n^2(\xi, x)}} 
\]  

(1)
\textbf{In[2]}: \quad \textbf{With}\{\varepsilon = 0.01, \ n = 32, \ \xi = 1.0001\},\ \\
\text{Plot}\left[\frac{1}{1 + \varepsilon^2 \text{EllipticRationalFunction}[n, \xi, x]^2},\ \\
\{x, 0, 2\}, \text{PlotStyle} \rightarrow \{\text{Thickness}[0.007]\}\right]\]

From \textbf{In[2]}:

Chebyshev polynomials, for the same order, give a far inferior approximation (thin line) of the square pulse as shown in the following figure.

\textbf{In[3]}: \quad \textbf{With}\{\varepsilon = 0.01, \ n = 32, \ \xi = 1.0001\},\ \\
\text{Plot}\left[\left\{\frac{1}{1 + \varepsilon^2 \text{EllipticRationalFunction}[n, \xi, x]^2},\ \\
\frac{1}{1 + \varepsilon^2 \text{ChebyshevT}[n, x]^2}\right\}, \{x, 0.95, 1.05\}, \ \\
\text{PlotStyle} \rightarrow \{\text{Thickness}[0.007], \text{Thickness}[0.002]\}, \ \\
\text{AxesOrigin} \rightarrow \{0.95, 0\}\right]\]

From \textbf{In[3]}:

Elliptic rational functions contain the one free parameter \(\xi\) that is used to adjust the slope of the pulse approximation. Chebyshev polynomials have no such parameter.

The pulse approximation with elliptic rational functions has many important applications in analog and digital signal processing and system design. Symbolic computation of elliptic rational functions and powerful symbolic algebra environments such as \textit{Mathematica}, made many successful industrial designs possible [4, 5, 6, 7, 8]. These designs produced robust systems (such as analog and digital filters), shortened time to market, and helped designers make cost-effective solutions.
Definition

We define the elliptic rational function in terms of the Jacobi elliptic functions as

\[ R_n(\xi, x) = \text{cd} \left( n \frac{K\left(\frac{1}{L_n(\xi)}\right)}{K\left(\frac{1}{\xi}\right)} \text{cd}^{-1}\left( x, \frac{1}{\xi} \right), \frac{1}{L_n(\xi)} \right), \]  

(2)

where \( \text{cd} \) is one of the 12 Jacobi elliptic functions (see \texttt{JacobiCD} in [9]), \( \text{cd}^{-1} \) is the inverse \( \text{cd} \) Jacobi elliptic function (see \texttt{InverseJacobiCD} in [9]), \( K \) is the complete elliptic integral of the first kind (see \texttt{EllipticK} in [9]), \( n \) is the order (a positive integer), \( \xi \) is the selectivity factor (\( \xi > 1 \)), and \( L_n(\xi) \) is the discrimination factor defined in the Introduction. It can also be defined as the value of the elliptic rational function for \( x = \xi \), that is, \( L_n(\xi) = R_n(\xi, \xi) \) [4].

\( R_n(\xi, x) \) can be represented by using the parametric equations

\[ R_n(\xi, x) = \text{cd} \left( n \omega K\left(\frac{1}{L_n(\xi)}\right), \frac{1}{L_n(\xi)} \right) \]

\[ x = \text{cd} \left( \omega K\left(\frac{1}{\xi}\right), \frac{1}{\xi} \right) \]

(3)

where \( \omega \) is an intermediate variable [4].

Traditionally, \( R_n(\xi, x) \) for known \( n, \xi, \) and \( x \), can be computed as follows.

1. Find \( K(1/\xi) \) using \texttt{EllipticK}.
2. Find \( \omega \) from the inverse \( \text{cd} \) Jacobi elliptic function.
3. Determine \( K(1/L_n(\xi)) \) and \( L_n(\xi) \) from the degree equation

\[ n \frac{K\left(\frac{1}{L_n(\xi)}\right)}{K\left(\sqrt{1 - \frac{1}{L_n(\xi)^2}}\right)} = \frac{K\left(\frac{1}{\xi}\right)}{K\left(\sqrt{1 - \frac{1}{\xi^2}}\right)}. \]

(4)

4. Find \( R_n(\xi, x) \) using the \( \text{cd} \) Jacobi elliptic function.

The Chebyshev polynomial \( T_n(x) \) can be derived from \( R_n(\xi, x) \) for \( \xi \to \infty \) because \( \text{cd}(\omega, 0) = \cos(\omega) \), that is,

\[ \lim_{\xi \to \infty} (R_n(\xi, x)) = \cos(n \cos^{-1}(x)) = T_n(x). \]

(5)

The elliptic rational function in terms of the Jacobi elliptic functions can be expanded as a rational function in terms of \( x \) [4]. The explicit formulas of the first three functions are

\[ R_1(\xi, x) = x \]

\[ R_2(\xi, x) = \frac{\left(1 + \sqrt{1 - \frac{1}{\xi^2}}\right)x^2 - 1}{\left(-1 + \sqrt{1 - \frac{1}{\xi^2}}\right)x^2 + 1} \]

(6)

(7)
\[ R_3(\xi, x) = x \frac{(1 + \text{dn}^2(\frac{2}{3} K(\frac{1}{\xi}), \frac{1}{\xi^2}))^2 x^2 - (1 + 2 \text{dn}(\frac{2}{3} K(\frac{1}{\xi}), \frac{1}{\xi^2}))}{(-1 + \text{dn}^2(\frac{2}{3} K(\frac{1}{\xi}), \frac{1}{\xi^2})) x^2 + 1}. \]  

In Mathematica, \( R_3(\xi, x) \) can be represented as follows.

\[
\text{ln[4]} = \left( \left[ \left[ 1 + \text{JacobiDN}\left(\frac{2}{3} \text{EllipticK}\left(\frac{1}{\xi^2}\right)\right) \right] \right]^2 x^2 - \left(1 + 2 \text{JacobiDN}\left(\frac{2}{3} \text{EllipticK}\left(\frac{1}{\xi^2}\right)\right)\right) \right) / \left(1 + \left(1 + \text{JacobiDN}\left(\frac{2}{3} \text{EllipticK}\left(\frac{1}{\xi^2}\right)\right)\right)^2 x^2\right) \\
\text{TraditionalForm}
\]

\[
\text{Out[4]} = x \left( x^2 \left( \frac{\text{dn}(\frac{2}{3} K(\frac{1}{\xi^2}), \frac{1}{\xi^2}))+1\right)^2 - 2 \text{dn}(\frac{2}{3} K(\frac{1}{\xi^2}), \frac{1}{\xi^2}) - 1 \right) x^2 + 1
\]

Note that JacobiDN requires \( \frac{1}{\xi^2} \) instead of \( \frac{1}{\xi} \).

The meaning of the Jacobi elliptic function notations follows: \( \text{cd}(u, k) \) means \( k \) is the modulus, \( \text{cd}(u \mid m) \) means \( m = k^2 \) is the parameter. The most common notation uses the form with \( k \), but the elliptic functions are implemented in Mathematica using the form with \( m \) instead.

## Properties

We implemented \( R_n(\xi, x) \) in Mathematica as \( \text{EllipticRationalFunction[n, c, x]} \).

The main properties of \( R_n(\xi, x) \) follow.

- The squared function is even, \( R_n^2(\xi, -x) = R_n^2(\xi, x) \).
- Over the unit interval \(-1 \leq x \leq 1\), \( R_n(\xi, x) \) is equiripple and \( R_n^2(\xi, x) \leq 1 \).
- For \( x = 1 \), \( R_n(\xi, 1) = 1 \).
- \( R_n(\xi, x) \) is monotonic increasing: \( 1 < R_n(\xi, x) < R_n(\xi, \xi) \), in the interval \( 1 < x < \xi \).
- For \( x \gtrsim \xi > 1 \), \( | R_n(\xi, x) | \gtrsim L_n(\xi) = R_n(\xi, \xi) \).
- The key feature of \( R_n(\xi, x) \) is

\[
R_n(\xi, x) = \frac{R_n(\xi, \xi)}{R_n(\xi, \frac{\xi}{x})}.
\]
• The poles of $R_n(\xi, x)$ can be simply expressed in terms of the zeros of $R_m(\xi, x)$:

$$x_{\text{pole}} = \frac{\xi}{x_{\text{zero}}}.$$  \hspace{1cm} (10)

The next figure illustrates the equiripple property of $R_n(\xi, x)$.

\[\text{In[5]:=} \quad \text{With[\{\xi = 1.1, R = EllipticRationalFunction[n, \xi, x]\},}
\]
\[\text{Plot[Evaluate[Table[EllipticRationalFunction[n, \xi, x], \{n, 1, 5\}],}
\]
\[\text{\{x, -1, 1\}, AxesLabel \rightarrow \{x, R\},}
\]
\[\text{PlotStyle \rightarrow \{Dashing[\{0.04\}], Dashing[\{0.03\}],}
\]
\[\text{Dashing[\{0.02\}], Dashing[\{0.01\}], Dashing[\{\}\}\}}]
\]

\[\text{From In[5]:=} \quad R_n(\xi, x)
\]

\[\begin{array}{c}
\text{Nestling Property}
\end{array}
\]

Higher-order elliptic rational functions can be generated from lower-order functions by using the nesting property [4]

$$R_{np}(\xi, x) = R_m(R_p(\xi, x), R_p(\xi, x)).$$  \hspace{1cm} (11)

For example, $R_4(\xi, x) = R_2(R_2(\xi, x), R_2(\xi, x))$.

\[\text{In[6]:=} \quad \text{With[\{\xi = 2\},}
\]
\[\text{R4 = EllipticRationalFunction[2, EllipticRationalFunction[2, \xi, x],}
\]
\[\text{EllipticRationalFunction[2, \xi, x]];}]
\]
\[\text{R4 \// FullSimplify \// TraditionalForm}
\]
\[\text{N[R4 \// Together \// TraditionalForm}
\]

\[\text{Out[7]/TraditionalForm=}
\]
\[\frac{1 + 2\sqrt{-24 + 14\sqrt{3}} \cdot 1}{(2\sqrt{17} \cdot x^2 - 2) + 1}
\]

\[\text{Out[8]/TraditionalForm=}
\]
\[\frac{13.8743 x^4 - 14.373 x^2 + 1.99484}{0.017926 x^4 - 0.516636 x^2 + 1.99484}
\]

The corresponding nesting formula can be derived for the zeros and poles of $R_n(\xi, x)$ as shown in [4]. For $n = 2^i \cdot 3^j$ the zeros and poles of $R_n(\xi, x)$ can be expressed symbolically in terms of $\xi$ without using the Jacobi elliptic functions. Here is an example.
\[\text{In[9]}: \quad R2 = \text{EllipticRationalFunction}[2, \xi, x];\]
\[\text{zeros2} = \text{FullSimplify}[\text{Solve}[\text{Numerator}[R2] = 0, x], \xi > 1 && \xi \in \text{Reals}]\]
\[\text{Out[10]}: \quad \{\{x \rightarrow \frac{1}{\sqrt{1 + \sqrt{1 - \frac{1}{x^2}}}}\}, \{x \rightarrow \frac{1}{\sqrt{1 - \sqrt{1 - \frac{1}{x^2}}}}\}\}\]

\[\text{In[11]}: \quad \text{poles2} = \text{FullSimplify}[\text{Solve}[\text{Denominator}[R2] = 0, x], \xi > 1 && \xi \in \text{Reals}]\]
\[\text{Out[11]}: \quad \{\{x \rightarrow \frac{1}{\sqrt{1 + \sqrt{1 - \frac{1}{x^2}}}}\}, \{x \rightarrow \frac{1}{\sqrt{1 - \sqrt{1 - \frac{1}{x^2}}}}\}\}\]

For orders \(n \neq 2^i 3^j\), \(R_n(\xi, x)\) cannot be expressed symbolically without the Jacobi elliptic functions or the like.

\section*{Algorithm}

In this section we implement the elliptic rational functions in Mathematica.

\subsection*{Elliptic Rational Functions}

First, we define the first-order function \(R_1(\xi, x) = x\).

\[\text{In[12]}: \quad \text{EllipticRationalFunction}[1, \xi, x_] := x;\]

We use the nesting property and the closed-form expressions [4] for orders \(n = 2^i 3^j\).

\[\text{In[13]}: \quad \text{EllipticRationalFunction}[n_. \text{EvenQ}, \xi, x_] :=\]
\[\text{Module}[[\text{t}], \text{t} = \sqrt{1 - \frac{1}{\text{DiscriminationFactor}[\frac{n}{2}, \xi]^2}};\]
\[\frac{(t + 1) \text{EllipticRationalFunction}[\frac{n}{2}, \xi, x]^2 - 1}{(t - 1) \text{EllipticRationalFunction}[\frac{n}{2}, \xi, x]^2 + 1];\]

\[\text{In[14]}: \quad \text{EllipticRationalFunction}[n_. (\text{Mod}[n, 3] = 0 \&), \xi, x_] :=\]
\[\text{Module}[[\text{b, c, r, y}], \text{y} = 1 - \frac{2}{\text{DiscriminationFactor}[\frac{n}{3}, \xi]^2}];\]
\[\text{b} = (1 - y^2)^{1/3}; c = \sqrt{1 + b + b^2}; r = \frac{1}{2} \left(\frac{y}{c} + \sqrt{2 + b + 2 c - 1}\right);\]
\[\left\{\text{EllipticRationalFunction}[\frac{n}{3}, \xi, x]\right\}\]
\[\left\{(r + 1)^2 \text{EllipticRationalFunction}[\frac{n}{3}, \xi, x]^2 - (1 + 2 r)\right\}/\]
\[\left\{(r^2 - 1) \text{EllipticRationalFunction}[\frac{n}{3}, \xi, x]^2 + 1\right\};\]
We use the Jacobi elliptic function JacobiSN and the complete elliptic integral of the first kind EllipticK for orders \( n \neq 2^j 3^j \).

\begin{verbatim}
In[15]:= EllipticRationalFunction[n_Integer, \( \xi \), x_] :=
   EllipticRationalFunctionJacobi[n, \( \xi \), x];

In[16]:= EllipticRationalFunctionJacobi[n_Integer, \( \xi \), x_] :=
   Module[{p, R, t, z},
      z = EllipticRationalFunctionZeroJacobi[n, \[1/\xi]]; p = EllipticRationalFunctionPoleJacobi[n, \[1/\xi]];
      R = Times @@ (t - z) /. t -> x
      ];

\end{verbatim}

**Discrimination Factor**

The first-order discrimination factor equals the selectivity factor.

\begin{verbatim}
DiscriminationFactor[1, \( \xi \)] := \( \xi \);

In[18]:= DiscriminationFactor[n_Integer, \( \xi \)] :=
   Module[{b}, b = DiscriminationFactor[n/2, \( \xi \)];
   (b + \[2] b^2 - 1)\[2];

In[19]:= DiscriminationFactor[n_Integer, \( \xi \)] :=
   Module[{b, c, r, y},
      y = 1 - \[2] DiscriminationFactor[n/3, \( \xi \)];
      b = (1 - y^2)^\[1/3];
      c = \[2] \( \sqrt{1 + b^2} \); r = \[2] \( \sqrt{y + \sqrt{2 + b + 2 c} - 1} \);
      1/2 \( \frac{(1 + 2 r)^3}{(1 - r)^3} \sqrt{(1 + r)} \);

\end{verbatim}

We use the Jacobi elliptic function JacobiSN and the complete elliptic integral of the first kind EllipticK for orders \( n \neq 2^j 3^j \).

\begin{verbatim}
DiscriminationFactor[n_Integer, \( \xi \)] :=
   DiscriminationFactorJacobi[n, \( \xi \)];

\end{verbatim}

**Zeros and Poles**

The zeros and poles of \( R_n(\xi, x) \) for orders \( n \neq 2^j 3^j \) are computed in terms of the Jacobi elliptic function JacobiSN and the complete elliptic integral of the first kind EllipticK.
The faster than The implemented

Traditional

Closed form

Closed-form expressions for the zeros and poles of $R_{\alpha}(\xi, x)$ exist [4] for orders $n = 2^i \cdot 3^j$ and these formulas do not require Jacobi functions.

\section*{Traditional Forms}

\begin{verbatim}
In[24]:= EllipticRationalFunctionZeroJacobi[n_Integer, k_] := Module[{t, z},
  t = \left(\frac{\text{JacobiSN}[\text{EllipticK}[k^2] + \frac{\text{EllipticK}[k^2]}{n} (2 \#1 - 1), k^2]}{\text{Range}[\text{Floor}[\frac{n}{2}]]}; \text{If}[\text{EvenQ}[n],
    z = \text{Join}[-t, \text{Reverse}[t]], z = \text{Join}[-t, \{0\}, \text{Reverse}[t]]; z];
  \right) /@ \text{Join}[-\text{Reverse}[t], t];

In[23]:= EllipticRationalFunctionPoleJacobi[n_Integer, k_] := Module[{t},
  t = \left(1/\left(k \text{JacobiSN}[\text{EllipticK}[k^2] + \frac{\text{EllipticK}[k^2]}{n} (2 \#1 - 1), k^2]\right)\right) /@ \text{Range}[\text{Floor}[\frac{n}{2}]]; \text{Join}[-\text{Reverse}[t], t];

\end{verbatim}

\section*{Timing}

We time how long it takes to calculate $R_{\alpha}(\xi, x)$ using our symbolic algorithm, EllipticRationalFunction[n,\xi, x], and the numeric algorithm [1] that we implemented in Mathematica as EllipticRationalFunctionJacobi[n,\xi, x]. The following figure shows that the proposed algorithm (solid line) runs much faster than the traditional algorithm for orders $n = 2^i \cdot 3^j$.

\begin{verbatim}
In[26]:= With[\{\xi = N[11, 10], t1List = \{\}; t2List = \{\};
  Do[t1 = \text{Timing}[\text{N}[\text{EllipticRationalFunction}[n, \xi, \text{x}]]];
    t2 = \text{Timing}[\text{N}[\text{EllipticRationalFunctionJacobi}[n, \xi, \text{x}]]];
    \text{AppendTo}[t1List, t1[[1]]]; \text{AppendTo}[t2List, t2[[1]]], \{n, 1, 32\}\}
\end{verbatim}
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References

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Miroslav D. Lutovac is a chief scientist at the Institute for Telecommunications and Electronics and a professor at the University of Belgrade, Serbia. His research interests include the theory and implementation of analog and digital signal processing, and the symbolic analysis and synthesis of multiplierless and multirate digital systems.

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Lutovac and Tosic are coauthors (with Brian Evans) of the book Filter Design for Signal Processing Using MATLAB and Mathematica [4]. They have developed SchematicSolver, a Mathematica application for mouse-driven interactive drawing of systems and for solving and implementing said systems.

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